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SELECTIONS
ILLUSTRATING THE HISTORY OF
**GREEK
MATHEMATICS**

WITH AN ENGLISH TRANSLATION BY
IVOR THOMAS

I
FROM THALES TO EUCLID



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VIII. HIPPOCRATES OF CHIOS

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(a) GENERAL

Philop. in *Phys.* A 2 (Aristot. 185 a 16), ed. Vitelli
31. 3-9

Ἴπποκράτης Χίος τις ὦν ἔμπορος, ληστρικῇ νηὶ περιπεσὼν καὶ πάντα ἀπολέσας, ἦλθεν Ἀθήνας γραφόμενος τοὺς ληστὰς, καὶ πολὺν παραμένων ἐν Ἀθήναις διὰ τὴν γραφὴν χρόνον, ἐφοίτησεν εἰς φιλοσόφους, καὶ εἰς τοσοῦτον ἕξεως γεωμετρικῆς ἦλθεν, ὡς ἐπιχειρῆσαι εὐρεῖν τὸν κύκλου τετραγωνισμόν. καὶ αὐτὸν μὲν οὐχ εὔρε, τετραγωνίσας δὲ τὸν μηνίσκον ᾤθη ψευδῶς ἐκ τούτου καὶ τὸν κύκλον τετραγωνίζειν· ἐκ γὰρ τοῦ τετραγωνισμοῦ τοῦ μηνίσκου καὶ τὸν τοῦ κύκλου τετραγωνισμόν ᾤθη συλλογίζεσθαι.

(b) QUADRATURE OF LUNES

Simpl. in *Phys.* A 2 (Aristot. 185 a 14), ed. Diels
60. 22-68. 32

Ὁ μὲντοι Εὐδήμος ἐν τῇ Γεωμετρικῇ ἱστορίᾳ οὐκ ἐπὶ τετραγωνικῆς πλευρᾶς δεῖξαι φησι τὸν Ἴπποκράτην τὸν τοῦ μηνίσκου τετραγωνισμόν,
234

VIII. HIPPOCRATES OF CHIOS

(a) GENERAL

Philoponus, *Commentary on Aristotle's Physics A 2*
(185 a 16), ed. Vitelli 31. 3-9

HIPPOCRATES of Chios was a merchant who fell in with a piratē ship and lost all his possessions. He came to Athens to prosecute the pirates and, staying a long time in Athens by reason of the indictment, consorted with philosophers, and reached such proficiency in geometry that he tried to effect the quadrature of the circle. He did not discover this, but having squared the lune he falsely thought from this that he could square the circle also. For he thought that from the quadrature of the lune the quadrature of the circle also could be calculated.*

(b) QUADRATURE OF LUNES

Simplicius, *Commentary on Aristotle's Physics A 2*
(185 a 14), ed. Diels 60. 22-68. 32

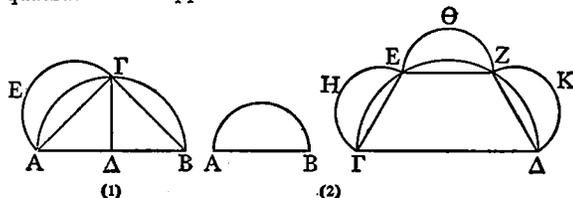
Eudemus, however, in his *History of Geometry* says that Hippocrates did not demonstrate the quadrature

* A lune (meniscus) is the figure included between two intersecting arcs of circles. It is unlikely that Hippocrates himself thought he had squared the circle, but for a discussion of this point see *infra*, p. 310 n. b.

ἀλλὰ καθόλου, ὡς ἂν τις εἴποι. εἰ γὰρ πᾶς μηνίσκος τὴν ἐκτὸς περιφέρειαν ἢ ἴσην ἔχει ἡμικυκλίου ἢ μείζονα ἢ ἐλάττονα, τετραγωνίζει δὲ ὁ Ἴπποκράτης καὶ τὸν ἴσην ἡμικυκλίου ἔχοντα καὶ τὸν μείζονα καὶ τὸν ἐλάττονα, καθόλου ἂν εἴη δεδειχῶς ὡς δοκεῖ. ἐκθήσομαι δὲ τὰ ὑπὸ τοῦ Εὐδήμου κατὰ λέξιν λεγόμενα ὀλίγα τιὰ προστιθεὶς (εἰς)¹ σαφήνειαν ἀπὸ τῆς τῶν Εὐκλείδου Στοιχείων ἀναμνήσεως διὰ τὸν ὑπομνηματικὸν τρόπον τοῦ Εὐδήμου κατὰ τὸ ἀρχαῖκόν ἔθος συντόμους ἐκθεμένου τὰς ἀποδόσεις. λέγει δὲ ὧδε ἐν τῷ δευτέρῳ βιβλίῳ τῆς Γεωμετρικῆς ἱστορίας.

¹ eis add. Usener.

^a As Alexander asserted. Alexander, as quoted by Simplicius in *Phys.* (ed. Diels 56. 1-57. 24), attributes two quadratures to Hippocrates.



In the first, AB is the diameter of a circle, AΓ, ΓB are sides of a square inscribed in it, and AEG is a semicircle described on AΓ. Alexander shows that

$$\text{lune AEG} = \text{triangle AΓΔ.}$$

In the second, AB is the diameter of semicircle and on ΓΔ, equal to twice AB, a semicircle is described. ΓE, EZ, ZΔ are sides of a regular hexagon, and ΓHE, EΘZ, ZKA are semicircles described on ΓE, EZ, ZΔ. Alexander shows that

of the lune on the side of a square^a but generally, as one might say. For every lune has an outer circumference equal to a semicircle or greater or less, and if Hippocrates squared the lune having an outer circumference equal to a semicircle and greater and less, the quadrature would appear to be proved generally. I shall set out what Eudemus wrote word for word, adding only for the sake of clearness a few things taken from Euclid's *Elements* on account of the summary style of Eudemus, who set out his proofs in abridged form in conformity with the ancient practice. He writes thus in the second book of the *History of Geometry*.^b

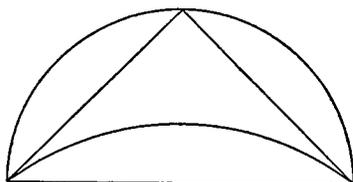
$$\text{lune GHE} + \text{lune EΘZ} + \text{lune ZKA} + \text{semicircle AB} = \text{trapezium ΓEZA.}$$

The proofs are easy. Alexander goes on to say that if the rectilinear figure equal to the three lunes ("for a rectilinear figure was proved equal to a lune") is subtracted, the circle will be squared. The fallacy is obvious and Hippocrates could hardly have committed it. This throws some doubt on the whole of Alexander's account, and Simplicius himself observes that Eudemus's account is to be preferred as he was "nearer to the times" of Hippocrates.

^b It is not always easy to distinguish what Eudemus wrote and what Simplicius has added. The task was first attempted by Allman (*Hermathena* iv., pp. 180-228; *Greek Geometry from Thales to Euclid*, pp. 64-75). Diels, in his edition of Simplicius published in 1882, with the help of Usener, printed in spaced type what they attributed to Eudemus. In 1883 Tannery (*Mémoires scientifiques* i., pp. 339-370) edited what he thought the Eudemian passages. Heiberg (*Philologus* xliii., pp. 336-344) gave his views in 1884. Rudio discussed the question exhaustively in 1907 (*Der Bericht des Simplicius über die Quadraturen des Antiphon und Hippokrates*), but unfortunately his judgement is not always trustworthy. Heath (*H.G.M.* i. 183-200) has an excellent analysis. In the following pages I have given only such passages as can safely be attributed to Eudemus and omitted the rest.

“Καὶ οἱ τῶν μηνίσκων δὲ τετραγωνισμοὶ δόξαντες εἶναι τῶν οὐκ ἐπιπολαίων διαγραμμάτων διὰ τὴν οἰκειότητα τὴν πρὸς τὸν κύκλον ὑφ’ Ἱπποκράτους ἐγράφησάν τε πρῶτον καὶ κατὰ τρόπον ἔδοξαν ἀποδοθῆναι· διόπερ ἐπὶ πλέον ἀψώμεθά τε καὶ διέλθωμεν. ἀρχὴν μὲν οὖν ἐποιήσατο καὶ πρῶτον ἔθετο τῶν πρὸς αὐτοὺς χρησίμων, ὅτι τὸν αὐτὸν λόγον ἔχει τὰ τε ὅμοια τῶν κύκλων τμήματα πρὸς ἄλληλα καὶ αἱ βάσεις αὐτῶν δυνάμει. τοῦτο δὲ ἐδείκνυεν ἐκ τοῦ τὰς διαμέτρους δεῖξαι τὸν αὐτὸν λόγον ἐχούσας δυνάμει τοῖς κύκλοις.

“Δειχθέντος δὲ αὐτῷ τούτου πρῶτον μὲν ἔγραφε μηνίσκου τὴν ἐκτὸς περιφέρειαν ἔχοντος ἡμικυκλίου



τίνα τρόπον γένοιτο ἂν τετραγωνισμός. ἀπεδίδου δὲ τοῦτο περὶ τρίγωνον ὀρθογώνιον τε καὶ ἰσοσκελές ἡμικύκλιον περιγράψας καὶ περὶ τὴν βάσιν τμήμα κύκλου τοῖς ὑπὸ τῶν ἐπιζευχθειῶν ἀφαιρουμένοις ὅμοιον. ὄντος δὲ τοῦ περὶ τὴν βάσιν τμήματος ἴσου τοῖς περὶ τὰς ἐτέρας ἀμφοτέροις, καὶ κοινοῦ προστεθέντος τοῦ μέρους τοῦ τριγώνου τοῦ ὑπὲρ τὸ τμήμα τὸ περὶ τὴν βάσιν, ἴσος ἔσται ὁ μηνίσκος τῷ τριγώνῳ. ἴσος οὖν ὁ μηνίσκος τῷ τριγώνῳ δειχθεὶς τετραγωνίζοιτο ἂν. οὕτως μὲν

“The quadratures of lunes, which seemed to belong to an uncommon class of propositions by reason of the close relationship to the circle, were first investigated by Hippocrates, and seemed to be set out in correct form; therefore we shall deal with them at length and go through them. He made his starting-point, and set out as the first of the theorems useful to his purpose, that similar segments of circles have the same ratios as the squares on their bases.^a And this he proved by showing that the squares on the diameters have the same ratios as the circles.^b

“Having first shown this he described in what way it was possible to square a lune whose outer circumference was a semicircle. He did this by circumscribing about a right-angled isosceles triangle a semicircle and about the base a segment of a circle similar to those cut off by the sides.^c Since the segment about the base is equal to the sum of those about the sides, it follows that when the part of the triangle above the segment about the base is added to both the lune will be equal to the triangle. Therefore the lune, having been proved equal to the triangle, can be squared. In this way, taking

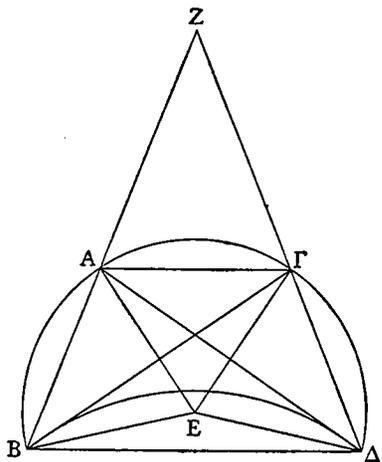
^a Lit. “as the bases in square.”

^b This is Eucl. xii. 2 (see *infra*, pp. 458-465). Euclid proves it by a method of exhaustion, based on a lemma or its equivalent which, on the evidence of Archimedes himself, can safely be attributed to Eudoxus. We are not told how Hippocrates effected the proof.

^c As Simplicius notes, this is the problem of Eucl. iii. 33 and involves the knowledge that similar segments contain equal angles.

οὖν ἡμικυκλίου τὴν ἔξω τοῦ μηνίσκου περιφέρειαν ὑποθέμενος ἐτετραγώνισεν ὁ Ἱπποκράτης τὸν μηνίσκον εὐκόλως.

“Ἐἴτα ἐφεξῆς μείζονα ἡμικυκλίου ὑποτίθεται συστησάμενος τραπέζιον τὰς μὲν τρεῖς ἔχον πλευρὰς



ἴσας ἀλλήλαις, τὴν δὲ μίαν τὴν μείζω τῶν παραλλήλων τριπλασίαν ἐκείνων ἐκάστης δυνάμει, καὶ τότε τραπέζιον περιλαβὼν κύκλῳ καὶ περὶ τὴν μεγίστην αὐτοῦ πλευρὰν ὅμοιον τμήμα περιγράφας τοῖς ὑπὸ τῶν ἴσων τριῶν ἀποτεμνομένοις ἀπὸ τοῦ κύκλου. ὅτι δὲ μείζον ἐστὶν ἡμικυκλίου τὸ λεχθὲν τμήμα, δῆλον ἀχθείσης ἐν τῷ τραπέζιῳ διαμέτρου. ἀνάγκη γὰρ ταύτην ὑπὸ δύο πλευρὰς ὑποτείνουσαν τοῦ τραπέζιου τῆς ὑπολοίπου μιᾶς μείζονα ἢ δι-

a semicircle as the outer circumference of the lune, Hippocrates readily squared the lune.

“Next in order he assumes [an outer circumference] greater than a semicircle [obtained by] constructing a trapezium having three sides equal to one another while one, the greater of the parallel sides, is such that the square on it is three times the square on each of those sides, and then comprehending the trapezium in a circle and circumscribing about ^a its greatest side a segment similar to those cut off from the circle by the three equal sides.^b That the said segment ^c is greater than a semicircle is clear if a diagonal is drawn in the trapezium. For this diagonal, subtending two sides of the trapezium, must be such that the square on it is greater than double the square on

^a *i.e.* “describing on.”

^b Simplicius here inserts a proof that a circle can be described about the trapezium.

^c *i.e.*, the segment bounded by the outer circumference. Eudemus is going to show that the angle in it is acute and therefore the segment is greater than a semicircle.

δυνάμει. ἡ δὲ ἐφ' ἧ ΕΗ ἤχθω παρὰ τὴν ἐφ' ἧ ΑΒ. καὶ ἀπὸ τοῦ Κ ἐπεζεύχθωσαν ἐπὶ τὰ Ε, Ζ. συμπίπτει δὲ ἐκβαλλομένη ἡ ἐπὶ τὸ Ζ ἐπιζευχθεῖσα τῇ ἐφ' ἧ ΕΗ κατὰ τὸ Η καὶ πάλιν ἀπὸ τοῦ Β ἐπὶ τὰ Ζ, Η ἐπεζεύχθωσαν. φανερόν δὲ ὅτι ἡ μὲν ἐφ' ἧ ΕΖ ἐκβαλλομένη ἐπὶ τὸ Β πεσεῖται (ὑπόκειται γὰρ ἡ ΕΖ ἐπὶ τὸ Β νεύουσα), ἡ δὲ ἐφ' ἧ ΒΗ ἴση ἔσται τῇ ἐφ' ἧ ΕΚ.

“Τούτων οὖν οὕτως ἐχόντων τὸ τραπέζιον φημι ἐφ' οὗ ΕΚΒΗ περιλήφεται κύκλος.

“Περιγεγράφθω¹ δὴ περὶ τὸ ΕΖΗ τρίγωνον τμήμα κύκλου, δηλὸν ὅτι ἐκάτερον τῶν ΕΖ, ΖΗ ὁμοιον ἐκάστῳ τῶν ΕΚ, ΚΒ, ΒΗ τμημάτων.

“Τούτων οὕτως ἐχόντων ὁ γενόμενος μηνίσκος οὐ ἐκτὸς περιφέρεια ἡ ΕΚΒΗ ἴσος ἔσται τῷ εὐθυγράμμῳ τῷ συγκειμένῳ ἐκ τῶν τριῶν τριγώνων τῶν ΒΖΗ, ΒΖΚ, ΕΚΖ. τὰ γὰρ ἀπὸ τῶν εὐθειῶν ἐφ' αἷς ΕΖ, ΖΗ ἀφαιρούμενα ἐντὸς τοῦ μηνίσκου ἀπὸ τοῦ εὐθυγράμμου τμήματα ἴσα ἔστι τοῖς ἐκτὸς

¹ Περιγεγράφθω... τμημάτων. In the text of Simplicius this sentence precedes the one above and Simplicius's comments thereon. It is here restored to the place which it must have occupied in Eudemus's *History*.

^a This is the first example we have had to record of the type of construction known to the Greeks as *νεύσεις*, *inclinations* or *vergings*. The general problem is to place a straight line so as to *verge towards* (pass through) a given point and so that a given length is intercepted on it by other lines. In this case the problem amounts to finding a length x such that, if Z be taken on $\Gamma\Delta$ so that $BZ=x$ and BZ be produced to

times the square on one of the radii.^a Let EH be drawn parallel to AB , and from K let [straight lines] be drawn joining E and Z . Let the straight line [KZ] joined to Z and produced meet EH at H , and again let [straight lines] be drawn from B joining Z and H . It is then manifest that EZ produced will pass through B —for by hypothesis EZ verges towards B —and BH will be equal to EK .

“This being so, I say that the trapezium $EKBH$ can be comprehended in a circle.

“Next let a segment of a circle be circumscribed about the triangle EZH ; then clearly each of the segments on EZ , ZH will be similar to the segments on EK , KB , BH .

“This being so, the lune so formed, whose outer circumference is $EKBH$, will be equal to the rectilinear figure composed of the three triangles BZH , BZK , EKZ . For the segments cut off from the rectilinear figure, inside the lune, by the straight lines EZ , ZH are (together) equal to the segments outside

meet the circumference in E , then $EZ^2 = \frac{2}{3}AK^2$, or $EZ = \sqrt{\frac{2}{3}}AK$. If this is done, $EB \cdot BZ = AB \cdot B\Gamma = AK^2$

or $(x + \sqrt{\frac{2}{3}}a) \cdot x = a^2$, where $AK = a$.

In other words, the problem amounts to solving the quadric equation

$$x^2 + \sqrt{\frac{2}{3}}ax = a^2.$$

This would be recognized by the Greeks as the problem of “applying to a straight line of length $\sqrt{\frac{2}{3}} \cdot a$, a rectangle exceeding by a square figure and equal in area to a^2 ,” and could have been solved theoretically by the Pythagorean method preserved in *Eucl. ii. 6*. Was this the method used by Hippocrates? Though it may have been, the authorities prefer to believe he used mechanical means (*H.G.M. i. 196*, Rudio, *loc cit.*, p. 59, Zeuthen, *Geschichte d. Math.*, p. 80). He could have marked on a ruler a length equal to $\sqrt{\frac{2}{3}}AK$ and moved it about until it was in the required position.

τοῦ εὐθυγράμμου τμημάτων ἀφαιρουμένοις ὑπὸ τῶν EK, KB, BH. ἐκάτερον γὰρ τῶν ἐντὸς ἡμιόλιόν ἐστιν ἐκάστου τῶν ἐκτὸς. ἡμιολία γὰρ¹ ὑπόκειται ἢ EZ τῆς ἐκ τοῦ κέντρου, τουτέστι τῆς EK καὶ KB καὶ BH. εἰ οὖν ὁ μὲν μηνίσκος τὰ τρία τμήματά ἐστι καὶ τοῦ εὐθυγράμμου τὸ παρὰ τὰ δύο τμήματα, τὸ δὲ εὐθύγραμμον μετὰ τῶν δύο τμημάτων ἐστὶ χωρὶς τῶν τριῶν, ἐστὶ δὲ τὰ δύο τμήματα τοῖς τρισὶν ἴσα, ἴσος ἂν εἴη ὁ μηνίσκος τῷ εὐθυγράμμῳ.

“Ὅτι δὲ οὗτος ὁ μηνίσκος ἐλάττονα ἡμικυκλίου τὴν ἐκτὸς ἔχει περιφέρειαν, δείκνυσι διὰ τοῦ τῆν EKH γωνίαν ἐν τῷ ἐκτὸς οὖσαν τμήματι ἀμβλείαν εἶναι. ὅτι δε ἀμβλεία ἐστὶν ἢ ὑπὸ EKH γωνία, δείκνυσιν οὕτως· ἐπεὶ² ἢ μὲν ἐφ’ ἢ EZ ἡμιολία ἐστὶ τῶν ἐκ τοῦ κέντρου δυνάμει, ἢ δὲ ἐφ’ ἢ KB μείζων τῆς ἐφ’ ἢ BZ ἢ διπλασία δυνάμει, φανερόν ὅτι καὶ ἢ ἐφ’ ἢ KE ἔσται τῆς ἐφ’ ἢ KZ ἄρα μείζων ἢ διπλασία δυνάμει. ἢ δὲ ἐφ’ ἢ EZ μείζων ἐστὶ δυνάμει τῶν ἐφ’ αἰς EK, KZ. ἀμβλεία ἄρα ἐστὶν ἢ πρὸς τῷ K γωνία, ἐλάττον ἄρα ἡμικυκλίου τὸ τμήμα ἐν ᾧ ἐστὶν.

“Οὕτως μὲν οὖν ὁ Ἱπποκράτης πάντα μηνίσκον ἐτετραγώνισεν, εἴπερ καὶ τὸν ἡμικυκλίου καὶ τὸν

¹ δυνάμει must be understood after ἡμιολία γὰρ, as Bretschneider first pointed out, but Diels and Rudio think that Simplicius probably omitted it as obvious, here and in his own comments.

² ἐπεὶ . . . ἐστὶν. Eudemus purports to give the proof in Hippocrates' own words. Unfortunately Simplicius's version is too confused to be worth reproducing. The proof is here given as reconstructed by Rudio. That it is substantially the proof given by Hippocrates is clear.

the rectilinear figure cut off by EK, KB, BH. For each of the inner segments is one-and-a-half times each of the outer, because, by hypothesis, the square on EZ is one-and-a-half times the square on the radius, that is, the square on EK or KB or BH. Inasmuch then as the lune is made up of the three segments and the rectilinear figure less the two segments—the rectilinear figure including the two segments but not the three—while the sum of the two segments is equal to the sum of the three, it follows that the lune is equal to the rectilinear figure.

“That this lune has its outer circumference less than a semicircle, he proves by means of the angle EKH in the outer segment being obtuse. And that the angle EKH is obtuse, he proves thus.

$$\text{Since} \quad EZ^2 = \frac{3}{2} EK^2$$

$$\text{and}^a \quad KB^2 > 2BZ^2,$$

it is manifest that $EK^2 > 2KZ^2$.

$$\text{Therefore} \quad EZ^2 > EK^2 + KZ^2.$$

The angle at K is therefore obtuse, so that the segment in which it is is less than a semicircle.

“Thus Hippocrates squared every lune, seeing that [he squared] not only the lune which has for its outer circumference a semicircle, but also the lune in which

^a This is assumed. Heath (*H.G.M.* i. 195) supplies the following proof:

By hypothesis, $EZ^2 = \frac{3}{2}KB^2$.

Also, since A, E, Z, Γ are concyclic,

$$EB \cdot BZ = AB \cdot B\Gamma = KB^2$$

$$\text{or} \quad EZ \cdot ZB + BZ^2 = KB^2 = \frac{2}{3}EZ^2.$$

It follows that $EZ > ZB$ and that $KB^2 > 2BZ^2$.

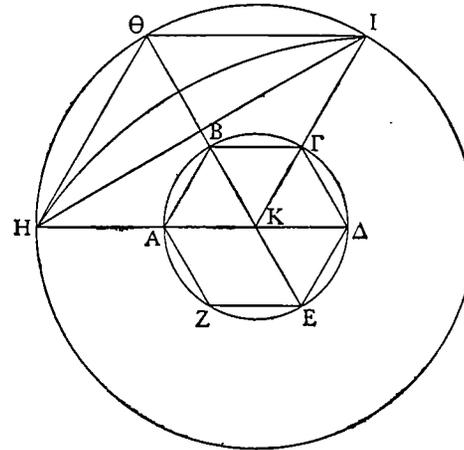
μείζονα ἡμικυκλίου καὶ τὸν ἐλάττονα ἔχοντα τὴν ἐκτὸς περιφέρειαν.

“ Ἀλλὰ μηνίσκον ἅμα καὶ κύκλον ἐτετραγώνισεν οὕτως· ἔστωσαν περὶ κέντρον ἐφ’ οὗ K δύο κύκλοι, ἡ δὲ τοῦ ἐκτὸς διάμετρος ἐξαπλασία δυνάμει τῆς τοῦ ἐντὸς καὶ ἑξαγώνου ἐγγραφέντος εἰς τὸν ἐντὸς κύκλον τοῦ ἐφ’ οὗ $ABΓΔEZ$ αἱ τε ἐφ’ ὧν $KA, KB, KΓ$ ἐκ τοῦ κέντρου ἐπιζευχθεῖσαι ἐκβεβλήσθωσαν ἕως τῆς τοῦ ἐκτὸς κύκλου περιφέρειας καὶ αἱ ἐφ’ ὧν $HΘ, ΘI, \langle HI \rangle^1$ ἐπεζεύχθωσαν καὶ δῆλον ὅτι καὶ αἱ $HΘ, ΘI$ ἑξαγώνου εἰσὶ πλευραὶ τοῦ εἰς τὸν μείζονα κύκλον ἐγγραφομένου. καὶ περὶ τὴν ἐφ’ ἣ HI τμήμα ὁμοιον τῷ ἀφαιρουμένῳ ὑπὸ τῆς ἐφ’ ἣ $HΘ$ περιγεγράφθω. ἐπεὶ οὖν τὴν μὲν ἐφ’ ἣ HI τριπλασίαν ἀνάγκη εἶναι δυνάμει τῆς ἐφ’ ἣ $ΘH$ τοῦ ἑξαγώνου πλευρᾶς (ἡ γὰρ ὑπὸ δύο τοῦ ἑξαγώνου πλευρᾶς ὑποτείνουσα μετὰ ἄλλης μιᾶς ὀρθὴν περι-

¹ HI add. Usener.

the outer circumference is greater, and that in which it is less, than a semicircle.

“ But he also squared a lune and a circle together in the following manner. Let there be two circles



with K as centre, such that the square on the diameter of the outer is six times the square on the diameter of the inner. Let a [regular] hexagon $ABΓΔEZ$ be inscribed in the inner circle, and let $KA, KB, KΓ$ be joined from the centre and produced as far as the circumference of the outer circle, and let $HΘ, ΘI, HI$ be joined. Then it is clear that $HΘ, ΘI$ are sides of a [regular] hexagon inscribed in the outer circle. About HI let a segment be circumscribed similar to the segment cut off by $HΘ$. Since then $HI^2 = 3ΘH^2$ (for the square on the line subtended by two sides of the hexagon, together with the square on one other

έχουσα γωνίαν τὴν ἐν ἡμικυκλίῳ ἴσον δύνανται τῇ διαμέτρῳ, ἢ δὲ διάμετρος τετραπλάσιον δύνανται τῆς τοῦ ἑξαγώνου ἴσης οὔσης τῇ ἐκ τοῦ κέντρου διὰ τὸ τὰ μήκει διπλάσια εἶναι δυνάμει τετραπλάσια), ἢ δὲ ΘΗ ἑξαπλασία τῆς ἐφ' ἢ AB, δῆλον ὅτι τὸ τμήμα τὸ περὶ τὴν ἐφ' ἢ ΗΙ περιγραφὴν ἴσον εἶναι συμβαίνει τοῖς τε ἀπὸ τοῦ ἐκτὸς κύκλου ὑπὸ τῶν ἐφ' αἰς ΗΘ, ΘΙ ἀφαιρουμένοις καὶ τοῖς ἀπὸ τοῦ ἐντὸς ὑπὸ τῶν τοῦ ἑξαγώνου πλευρῶν ἀπασῶν. ἢ γὰρ ΗΙ τῆς ΗΘ τριπλάσιον δύνανται, ἴσον δὲ τῇ ΗΘ δύνανται ἢ ΘΙ, δύνανται δὲ ἑκατέρα τούτων ἴσον καὶ αἱ ἕξ πλευραὶ τοῦ ἐντὸς ἑξαγώνου, διότι καὶ ἡ διάμετρος τοῦ ἐκτὸς κύκλου ἑξαπλάσιον ὑπόκειται δύνασθαι τῆς τοῦ ἐντὸς, ὥστε ὁ μὲν μηνίσκος ἐφ' οὗ ΗΘΙ τοῦ τριγώνου ἐλάττων ἂν εἴη ἐφ' οὗ τὰ αὐτὰ γράμματα τοῖς ὑπὸ τῶν τοῦ ἑξαγώνου πλευρῶν ἀφαιρουμένοις τμήμασιν ἀπὸ τοῦ ἐντὸς κύκλου. τὸ γὰρ ἐπὶ τῆς ΗΙ τμήμα ἴσον ἦν τοῖς τε ΗΘ, ΘΙ τμήμασι καὶ τοῖς ὑπὸ τοῦ ἑξαγώνου ἀφαιρουμένοις. τὰ οὖν ΗΘ, ΘΙ τμήματα ἐλάττω ἐστὶ τοῦ περὶ τὴν ΗΙ (τμήματος τοῖς)¹ τμήμασι [καὶ]² τοῖς ὑπὸ τοῦ ἑξαγώνου ἀφαιρουμένοις. κοινῶ οὖν προστεθέντος τοῦ ὑπὲρ τὸ τμήμα τὸ περὶ τὴν ΗΙ μέρους τοῦ τριγώνου, ἐκ μὲν τούτου καὶ τοῦ περὶ τὴν ΗΙ τμήματος τὸ τρίγωνον ἔσται, ἐκ δὲ τοῦ αὐτοῦ καὶ τῶν ΗΘ, ΘΙ τμημάτων ὁ μηνίσκος. ἔσται οὖν ἐλάττων ὁ μηνίσκος τοῦ τριγώνου τοῖς ὑπὸ τοῦ ἑξαγώνου ἀφαιρουμένοις τμήμασιν. ὁ ἄρα

¹ τμήματος τοῖς add. Bretschneider.

² καὶ om. Bretschneider.

^a If HA be a side of the hexagon, then IA is a diameter and the angle IHA is right. Therefore $HI^2 + HA^2 = IA^2$,
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side, is equal, since they form a right angle in the semicircle, to the square on the diameter, and the square on the diameter is four times the side of the hexagon, the diameter being twice the side in length and so four times as great in square ^a), and $\Theta H^2 = 6 AB^2$, it is manifest that the segment circumscribed about HI is equal to the segments cut off from the outer circle by HΘ, ΘΙ, together with the segments cut off from the inner circle by all the sides of the hexagon.^b For $HI^2 = 3 H\Theta^2$, and $\Theta I^2 = H\Theta^2$, while ΘI^2 and $H\Theta^2$ are each equal to the sum of the squares on the six sides of the inner hexagonal, since, by hypothesis, the diameter of the outer circle is six times that of the inner. Therefore the lune HΘΙ is smaller than the triangle HΘΙ by the segments taken away from the inner circle by the sides of the hexagon. For the segment on HI is equal to the sum of the segments on HΘ, ΘΙ and those taken away by the hexagon. Therefore the segments [on] HΘ, ΘΙ are less than the segment about HI by the segments taken away by the hexagon. If to both sides there is added the part of the triangle which is above the segment about HI,^c out of this and the segment about HI will be formed the triangle, while out of the latter and the segments [on] HΘ, ΘΙ will be formed the lune. Therefore the lune will be less than the triangle by the segments taken away by the hexagon. For the lune and the

and so $HI^2 + \Theta H^2 = IA^2 = 4\Theta H^2$ (since $IA = 2\Theta H$). Consequently $HI^2 = 3\Theta H^2$.

^b For (segment on HI) = 3 (segment on HΘ)
= 2 (segment on HΘ) + 6 (segment on AB)
= (segments on HΘ, ΘΙ) + (all segments of inner circle).

^c i.e., the figure bounded by HΘ, ΘΙ and the arc IH.

μηρίσκος καὶ τὰ ὑπὸ τοῦ ἑξαγώνου ἀφαιρούμενα τμήματα ἴσα ἔστιν τῷ τριγώνῳ. καὶ κοινὸν προστεθέντος τοῦ ἑξαγώνου τὸ τρίγωνον τοῦτο καὶ τὸ ἑξαγώνον ἴσα ἔστι τῷ τε μηρίσκῳ τῷ λεχθέντι καὶ τῷ κύκλῳ τῷ ἐντός. εἰ οὖν τὰ εἰρημένα εὐθύγραμμα δυνατόν τετραγωνισθῆναι, καὶ τὸν κύκλον ἄρα μετὰ τοῦ μηρίσκου.”

(c) TWO MEAN PROPORTIONALS

Procl. in *Eucl.* i., ed. Friedlein 212. 24-213. 11

Ἡ δὲ ἀπαγωγή μεταβάσις ἔστιν ἀπ’ ἄλλου προβλήματος ἢ θεωρήματος ἐπ’ ἄλλο, οὐ γνωσθέντος ἢ πορισθέντος καὶ τὸ προκείμενον ἔσται καταφανές, οἷον ὡσπερ καὶ τοῦ διπλασιασμοῦ τοῦ κύβου ζητηθέντος μετέθεσαν τὴν ζήτησιν εἰς ἄλλο, ὃ τοῦτο ἔπεται, τὴν εὔρεσιν τῶν δύο μέσων, καὶ τὸ λοιπὸν ἐζήτησαν, πῶς ἂν δύο δοθεισῶν εὐθειῶν δύο μέσαι ἀνάλογον εὔρεθειεν. πρῶτον δὲ φασὶ τῶν ἀπορουμένων διαγραμμάτων τὴν ἀπαγωγὴν ποιήσασθαι Ἰπποκράτην τὸν Χίον, ὃς καὶ μηρίσκον ἔτετραγώνισε καὶ ἄλλα πολλὰ κατὰ γεωμετρίαν εὔρεν εὐφύης περὶ τὰ διαγράμματα εἶπερ τις ἄλλος γενόμενος.

* What Hippocrates showed was that if $\frac{a}{x} = \frac{x}{y} = \frac{y}{b}$, then

segments taken away by the hexagon are equal to the triangle. When the hexagon is added to both sides, this triangle and the hexagon will be equal to the aforesaid lune and to the inner circle. If then the aforementioned rectilinear figures can be squared, so also can the circle with the lune.”

(c) TWO MEAN PROPORTIONALS

Proclus, on *Euclid* i., ed. Friedlein 212. 24-213. 11

Reduction is a transition from one problem or theorem to another, whose solution or construction makes manifest also that which is propounded, as when those who sought to double the cube transferred the investigation to another [problem] which it follows, the discovery of the two means, and from that time forward inquired how between two given straight lines two mean proportionals could be found. They say the first to effect the reduction of the difficult constructions was Hippocrates of Chios, who also squared a lune and discovered many other things in geometry, being unrivalled in the cleverness of his constructions.^a

$\frac{a^3}{x^3} = \frac{a}{b}$, so that if $b = 2a$, a cube of side x is twice the size of a cube of side a . For a fuller discussion, see *infra*, p. 258 n. b. It has been supposed from this passage that Hippocrates discovered the method of geometrical reduction, but this is unlikely.