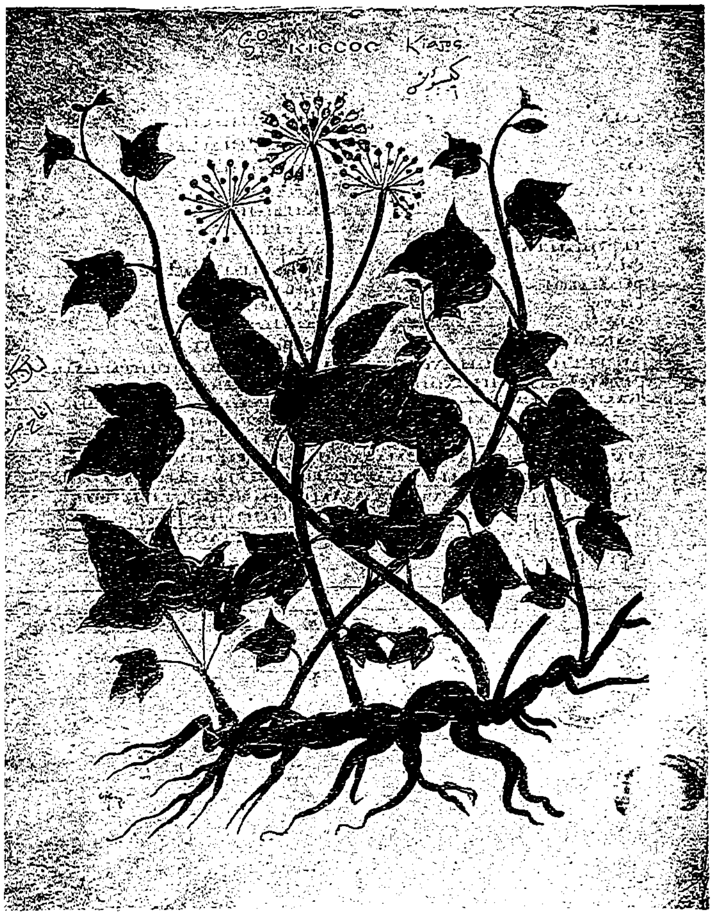

ıed within the medical encyclopedia of Dioscorides (fl. mid-first century ı is a survey of the plant species and their therapeutic value. The drawings · shown here are taken from the famous illustrated Byzantine manuscript of 'ork (made in 512 A.D.) and now held in the collection of ancient manu-s of the Austrian National Library (Cod. Vindobon. Med. Gr. 1, f. 174v). reproduction has been made from the facsimile edition of the Codex (Aka-ıche Druck- und Verlaganstalt, Graz, 1966–70, vol. 2), with the kind per-ɔn of the publishers and the Oesterreichische Nationalbibliothek. (For dis-ɔn, see chap. 6, sect. iv.)

The Ancient Tradition of Geometric Problems

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[19] The pun is possible in Greek, where the word *telos* and derivatives carry the various senses of "end," "end result," and "goal"; cf. Aristotle, *Physics*, II, 2, 194 a 30.

[20] On Eudoxus, see Chapter 3.

[21] The ancient field of the philosophy of mathematics is richer than my statement suggests. But that study seems primarily to have engaged philosophers, interested in accommodating mathematical questions within their own favored philosophical outlooks, like Platonism, Epicureanism, or Scepticism. The participation of mathematicians within this field appears to have been slight, and its influence on their researches slighter still. This ancient philosophical field is the subject of recent papers and a monograph in progress by I. Mueller.

[22] We have already alluded to an instance of this attitude, in the treatment of the two versions of the Delian story surviving from Eratosthenes.

[23] See, for instance, Carpus' remarks on the nature of geometric problems, cited in Chapter 8. Dissatisfaction with the ancient formal style led several 17th-century geometers to reconsider questions of method and notation; see, in particular, Descartes, *Géométrie*. The method of analysis thus served to stimulate the development of new algebraic approaches in geometry; cf. M. S. Mahoney, *The Mathematical Career of Pierre de Fermat*, Princeton University Press, 1973, Ch. II.1, III, V.

[24] This example is not hypothetical; cf. Chapter 8.

[25] See the view expressed by Enriques, cited in note 13 above.

[26] This is clearly the effect which von Wilamowitz' rejection of the Eratosthenes document has had, discouraging use of its potentially significant testimony to the early history of cube duplication; cf. note 10 above and Chapter 2.

[27] Pappus' repute both as a mathematician and as a spokesman on the nature of mathematics is an issue in Chapter 8 and will be further examined in the sequel volume.

[28] See Chapter 4.

CHAPTER 2

Beginnings and Early Efforts

When did geometers first initiate the quest for solutions of geometric problems? One seems to find some hints toward a form of answer from certain discussions by Proclus, who, as leader of the Academy in Athens in the 5th century A.D., included guidance in the study of elementary geometry as part of his teachings in the Platonic philosophy. His commentary on Euclid's Book I is a rich source not only of ancient views on the philosophical aspects relating to mathematics, but also on details of its historical development. From Proclus we receive several items on pre-Euclidean efforts derived from the history of geometry compiled by Aristotle's disciple Eudemus of Rhodes toward the close of the 4th century B.C. Two of these relate to problems of construction: I, 12 (how to draw the perpendicular to a given line through a given point not on that line) and I, 23 (how to draw an angle equal to a given angle at a given point on a given line). Reportedly, these problems were first solved by Oenopides of Chios, a geometer and astronomer, elsewhere identified as an associate of Hippocrates of Chios and hence datable to after the middle of the 5th century B.C.[1]

This witness to the work of Oenopides has led scholars to wonder how it could be that such elementary efforts emerged so late. Assuming to the contrary that they must have been known much earlier, they infer that it was the specific form of the solutions, as found in Euclid's treatment, which Oenopides originated, so that he would become the one responsible for the formal project of effecting geometric constructions within the restriction of employing compass and straightedge alone, as is characteristic of Euclid's method in Book I. Presumably, then, Oenopides had access to a diverse range of ways for solving these and other problems, and set about the task of regularizing and classifying problems according to the means of construction adopted.[2]

If such a bald statement of this thesis seems unsympathetic, a more subtle representation would merely camouflage its intrinsic implausibility. To be sure, much older precedents for the sort of geometric construction assigned to Oenopides can be detected. For instance, Babylonian tablets from the mid-2nd millennium B.C. display figures of circles and squares drawn with the assistance of instruments, while the Egyptians' use of cords in their practice of geometric mensuration earned for them the name "rope-stretchers" among the later Greeks.[3] But the type of inquiry supposedly engaged in by Oenopides is of an entirely different order of formal sophistication: to effect constructions on the consciously formal restriction to a specified set of means. It is no obvious matter to explain how and why such a distinctly formal move in geometry should appear so early. Attempts to account for it within the context of an alleged formalization of the general geometric field through linkage with pre-Socratic philosophical developments, as with the 5th-century Eleatics and Pythagoreans, are fraught with difficulties of their own.[4]

In the case of Oenopides, such doubts are magnified upon closer consideration of the details communicated by Proclus; specifically, that Oenopides introduced his construction of the perpendicular for its utility in astronomy; and that for the phrase "at right angles" he employed the term "gnomon-wise" (*kata gnōmona*), that is, with reference to the "gnomon" or pointer of a sundial. It thus seems evident that Eudemus, Proclus' source, has gleaned his information from a treatment of astronomical constructions known under the name of Oenopides.[5] But then it hardly follows that these were the earliest constructions of these problems, even under the special manner of construction they employ, but only that these were the earliest such instances of them known to Eudemus. More importantly, Oenopides would surely not have been occupied in a consciously formal geometric effort, for the astronomical context indicates that he was showing how to arrange the construction of astronomical devices, like sundials, via appropriate instruments of construction. The latter surely included compasses and rulers, but we have no reason to suppose that he rejected the use of others. Such devices as set squares, forms of compasses and sectors, angle-measuring devices, plummets, and sliding marked rulers were all in the repertory of techniques available to geometric practitioners in antiquity, and attested in the mathematical literature from the decades before and after Euclid.[6]

It is imperative, I maintain, to raise these doubts about the formal nature of the work of Oenopides and his contemporaries. While *we* might consider it obvious and natural to classify constructions according to the means employed and to assign privileged status to those demanding only compass and straightedge for their execution, our intuitions in such matters are thoroughly conditioned through knowledge and adoption of the objectives of the formal geometric tradition, advanced primarily through the works of Euclid and Apollonius. But in fact this formal restriction on the treatment of problems in itself betokens attainment of a sophisticated theoretical level. Thus, a historical inquiry into the study of problems must recognize the sophistication implicit in this move and ask about the time, manner, and motive of its introduction. Most treatments of ancient geometry, and specifically those dealing with the history of problem solving, assume this formal motive at the outset.[7] This assumption, I believe, presents serious obstacles against gaining insight into the development of the ancient studies as they actually happened. Let us then set aside that assumption. Through the examination of the evidence of the early work, specifically on the two problems of cube duplication and circle quadrature, we may hope to discover the roots of the ancient field of problem solving.

THE DUPLICATION OF THE CUBE

We possess accounts of the early stages of the study of the cube duplication in two reports derived from Eratosthenes of Cyrene, a prominent man of science and letters from the latter part of the 3rd century B.C. One of these is the fragment from a scene in his dialogue, the *Platonicus*, and is preserved by Theon of Smyrna and Plutarch in writings from the 2nd century A.D.[8] The other takes the form of a letter addressed by Eratosthenes to his royal patron Ptolemy III Euergetes in association with a transcript of the description of an actual model of his device, the "mesolabe" ("mean-taker"), for the mechanical solution of this problem and the text of an epigram dedicating it in the temple of the Ptolemies. The latter account is preserved as one of eleven texts on this problem compiled by Eutocius of Askalon, the 6th-century-A.D. commentator on Archimedes' work.[9] The value of such a document for examining the early history of these efforts would seem evident; but it has not been exploited by historians, owing to their general acceptance of the case challenging its authenticity.[10] It is thus important first to consider whether that case is to be sustained.

Eutocius' version runs to five full pages in the standard edition by J. L. Heiberg. We may divide it into five sections: (i) A historical introduction cites a scene from a tragedy about Minos by an unnamed dramatist and a legend telling of Plato's involvement with the oracle of Delos as precedents for an interest in the problem of doubling the cube. Specific note is made of efforts by Hippocrates of Chios and by three associates of Plato: Archytas, Eudoxus, and Menaechmus. (ii) Eratosthenes' mechanism for finding means is cited and praised for its greater practicality in comparison with the earlier methods. Several contexts of potential application are enumerated, among them ship building and military engineering, by way of illustrating the utility of the device. (iii) A full account of the geometric theory of the device is given. (iv) This is followed by a description of certain physical details pertaining to the actual construction of a working model. (v) The text closes with a description of the votive monument raised by Eratosthenes to commemorate its invention: this consisted of a bronze exemplar of the "mesolabe" set atop a pillar with an explanatory inscription engraved below. A full transcript of the latter is given, including (a) a brief account of the geometric theory of the device, parallel to that in (iii), and (b) an epigram of eighteen verses singing the praises of its virtues in contrast to the same prior efforts named in (i), and finally dedicating it to Ptolemy and his son.[11]

Since the penetrating examination of this document by U. von Wilamowitz-Moellendorff in 1894, the authenticity of the closing epigram and of the descriptive text just preceding it [(v-a) and (v-b)] have been granted.[12] The epigram is too expertly crafted, too fine a specimen of the difficult form of Hellenistic elegiac couplets, he argued, to be so easily dismissed as a forgery the way critics before him had contended. The descriptive text on the model of the device would raise no suspicions (save for an occasional trivial interpolation), while the event of an inventor or artisan's dedicating a work sample at a shrine is not without precedents in this period. But von Wilamowitz had no such approval to give to the accompanying letter. First, he argued, this presents an extended account of the device (iii) which would surely be superfluous, given the presence of the dedicatory model itself. Second, on points of detail it conflicts with the "genuine" part of the text (v), he alleged, as well as with the fragment from the *Platonicus* available to us through Theon and Plutarch; on the whole, however, it appears to assert little beyond what one might have read in the inscription or inferred from that. Third, this text was not known to Pappus, who wrote about two centuries before Eutocius, for his own version of Eratosthenes' method differs on points of detail in the construction and proof from the one given in the letter, although it presents a highly condensed version of materials parallel to Eutocius' items (iv) and (iii) or (v-a).[13] Finally, and most damaging of all, the letter is banal: it is written without any sense of style and without any recognition of the occasion of its communication. One can hardly conceive, argued von Wilamowitz, that a man of such literary talents as Eratosthenes, writing to his patron, should frame his account in such a pedestrian manner. Thus, he concluded, the letter must be taken as a late forgery, still unknown at the time of Pappus; its author presumably transcribed from the votive monument the epigram of Eratosthenes and the partially mutilated description of the device and then, misled by the epigram's devotional invocation of Ptolemy as deity of the shrine, mistook the offering as a personal gift and so produced the letter as a companion explanatory document. But, he added, the ineptness of the forger's effort is betrayed on many points, most strikingly in its omission of second person forms of address almost until the very end. On the other hand, the composition has preserved for us the genuine epigram, for which we may be grateful.

Shall one accept this view of the provenance of the letter? To be sure, its style is concise, perhaps a bit abrupt; but it is not an illiterate production. One has yet to display such outright anachronisms which would mark it as impossible for a writer of the 3rd century B.C. Moreover, the historical and technical information it presents is entirely compatible with what we learn from our other sources on the early studies. Indeed, the absence of anachronisms relating to the mathematical content is impressive and would suggest at the very least a degree of thoughtfulness and skill on the part of the alleged forger. A comparison with the "genuine" parts of the document shows that the "forger" has not limited himself to merely what has appeared in the epigram, but goes beyond it on several points in ways which indicate a real familiarity with this material, rather than mere fabrication. For instance, he observes that of the earlier methods only that of Menaechmus admitted any practical implementation "to a certain extent and that with difficulty" (*dyscherōs*); by contrast, the epigram refers only to the constructions by Archytas as "unwieldy" (*dysmēchana*), where in context the charge applies indifferently to the methods of both Menaechmus and Eudoxus as well.[14]

Relative to the argument concerning Pappus, no one has ever presumed that Pappus passed on *all* the sources at his disposal. Even if he did not possess a copy of the source in the form used by Eutocius, that hardly implies the nonexistence of that source at Pappus' time. After all, Eutocius himself presumably passed over the version in Pappus (if he had access to that) in favor of the text he actually reproduces. Surely, Pappus was capable of the same kind of editorial selectivity. As it happens, his text on Eratosthenes is rather more concise than that given by Eutocius, but on the whole not superior to it. Von Wilamowitz makes much of certain discrepancies (e.g., the shape of the sliding plates as triangular according to Pappus, but rectangular according to Eutocius)[15] which have no bearing on the feasibility of the design and its realization. But he is silent on certain insights of Eutocius omitted by Pappus: e.g., that the "mesolabe" can be used for finding not just two, but any number of mean proportionals by the insertion of additional plates.[16] The notion that Eutocius' report could result merely by transcribing and adapting what could be seen on the monument is simply not true, although it might well hold for the version in Pappus. In particular, Eutocius is far more detailed in his recommendations on the physical construction of the device. It is difficult enough to accept that the inscription should remain more or less intact as late as Pappus' time, early in the 4th century A.D.[17]; but that such a memorial of pagan worship could survive the volatile spirit of the later part of that century and the next is truly incredible.[18] Von Wilamowitz' argument, that most of the information in the letter would be superfluous in the presence of the monument itself, is not compelling. One may consider the example of the ornate astrolabe invented and built by Synesius, disciple of Hypatia, statesman, philosopher and bishop; the specimen was forwarded to his noble associate Paeonius accompanied by a letter describing in detail the conception and physical execution of the device. Many of these details would be evident from the device, including the two epigrams engraved on it, which Synesius quotes in full: "let it be set down for such as may read it later, since for you it is enough that it lies on the tablet."[18a] One thus sees in Synesius' dedication a close parallel to Eratosthenes' letter on the "mesolabe."

The strongest indication of the letter's authenticity, however, would appear to be precisely what von Wilamowitz took to be its most questionable feature: its banality. For surely a forger would have let his imagination run wider than the narrow compass of factual materials given here. By contrast, those facts do amount to the kind of information which technical accounts accompanying other mathematical writings of this period contain. The prefaces to the treatises by

Archimedes, for instance, rarely go beyond stating the principal theorems in the work and making brief observations on the methods used.[19] One would suppose that his correspondents Dositheus and Eratosthenes were receiving these communications by virtue of their official positions at Alexandria; but even this inference is not directly affirmed by anything actually said in the prefaces.[20] The *Sand-Reckoner* provides a good parallel to Eratosthenes' letter. It is devoted primarily to the exposition of the geometry and number theory needed for a specific computation dealing with the dimensions of the cosmos.[21] Within it are details of the design of a sighting device by Archimedes himself and a report of particulars, both geometric and observational, relating to its actual use. The discussion elaborates a theme introduced through a general literary reference (i.e., the inadequacy of the account of the infinite by certain unnamed philosophers). In all these respects, then, it follows a pattern like that of Eratosthenes' letter. Indeed, both are framed as communications to royal patrons, yet the *Sand-Reckoner*, despite its much greater length, is as sparing in its use of second-person forms as the letter is.[22] The Gelon, addressed as "king" by Archimedes, is believed to have served as regent in Syracuse for some thirty years, although he never reigned as king in his own right.[23] Was he still a youth or already a mature man when Archimedes wrote to him? Was the writing sent to him at Syracuse, or elsewhere? And where was Archimedes himself at that time: at home or abroad? Did he read the letter orally before an audience or was the writing intended for Gelon's personal study? Was Archimedes a kinsman of the regent or his former tutor? The document provides not a single clue toward the answer to any of these questions of context.

In view of this, how can one fault Eutocius' document for its failure to establish a setting? Von Wilamowitz' critique thus falters through his own failure to reckon with the character of the appropriate literary genre. Without grounds for suspicion, then, we may treat Eutocius' text no less seriously than any other, as a historical source. Indeed, von Wilamowitz' plea for admitting the epigram, falling within a genre of which he is a recognized master, would naturally prepare us for admitting the whole document as genuine. The burden of proof rests upon those who would maintain the contrary.

Under the view that the letter is authentic, its ostensible conflict with the alternative account from the *Platonicus* takes on entirely new significance. With reference to the efforts of the geometers in Plato's Academy, the letter says this:

> [a] After some time they say that certain Delians fell into the same difficulty as they set about to double one of the altars in accordance with an oracle, and so sending out word they asked the geometers with Plato in the Academy to find for them what was sought. These applied themselves diligently and sought (how) to take two means of two given lines, [b] and Archytas of Taras is said to have discovered (a solution) by means of semicylinders, Eudoxus by means of the so-called curved lines. [c] But it happened that these all wrote in demonstrative fashion, none being able to manage it in a practical way or to put it into use, save to a certain small extent Menaechmus and that with difficulty.[24]

It would appear that two or three different sources have been stitched together: (a) an account of the legend of the Delian oracle and the communication to the Academy; (b) a reference to solutions by Archytas and Eudoxus; and (c) an account somehow acquainted with writings derived from these geometers, in particular, from Menaechmus. Doubtless, communications between Archytas and the Academy were consistently good, through visits by Plato and others to Italy, and through correspondence; but the ancient biographical traditions do not mention an actual residence by Archytas at the Academy.[25] Thus, line (b) reads uncomfortably as a simple continuation of (a). Further, both (a) and (b) are reported at some detachment from primary sources ("they say;" "it is said"). By contrast, (c) seems to speak from a certain familiarity with such sources. When Eutocius elsewhere presents Archytas' method, Eudemus is named as source.[26] One would suppose that Eudemus also treated the methods of Eudoxus and Menaechmus and that his versions were available to Eratosthenes. That being so, the tentative tone of line (b) is puzzling: surely it would be *obvious* that Archytas used semicylinders and Eudoxus used curved lines. Here then is a place where an interpolator may have inserted a line, based on his reading of the allusions to these geometers in the epigram.[27] Without it, line (c) follows as a fully appropriate technical observation after the story in (a).

The other account of this incident is transmitted by Theon of Smyrna in the introductory section of his compilation of mathematical materials pertinent to the study of Plato:

> For Eratosthenes says in his writing the *Platonicus* that when the god pronounced to the Delians in the matter of deliverance from a plague that they construct an altar double of the one that existed, much bewilderment fell upon the builders who sought how one was to make a solid double of a solid. Then there arrived men to inquire of this from Plato. But he said to them that not for want of a double altar did the god prophesy this to the Delians, but to accuse and reproach the Greeks for neglecting mathematics and making little of geometry.[28]

This version of the story is further attested in two passages from Plutarch, for whom the moral is that one needs dialectical skill to maneuver among the ambiguities of oracles.[29] Following von Wilamowitz, many have taken these discrepancies to militate against the authenticity of the version in Eutocius. One is thus to suppose that the forger of the letter diluted Eratosthenes' account in the preparation of his own. But why should he have sought to remove all those details which give the story vividness: the occasion of the plague and Plato's personal intervention to interpret the true meaning of the oracle? The story in the letter seems to have no point beyond explaining why a certain group of geometers came to concern themselves with cube duplication.

The embellishments reported by Theon and Plutarch suggest that their versions derived from the elaboration of the more pedestrian account cited by Eutocius. Doubtless, Eratosthenes himself was responsible for these changes, as he sought in his *Platonicus* to dramatize aspects of Plato's view of mathematics. After all, motives of this very sort led Theon and Plutarch to cite this passage in their own works. Elsewhere, Plutarch exploits the cube duplication to illustrate another aspect of Plato's philosophy: that geometry is not a matter of perceptible things, but only of the eternal and incorporeal. Thus, he has Plato criticize the mechanical solutions of Archytas, Eudoxus, and Menaechmus as spoiling the true good of geometry.[30] This alternative version echoes the second part of Eutocius' account, where the practicability of these geometers' solutions is brought up for comment. But there is a patent difference: according to Eutocius, Eratosthenes criticizes the earlier efforts for their being too abstract; by contrast, Plutarch derives from him a criticism of their overly mechanical character. Of course, this discrepancy need only be one of emphasis. The surviving accounts of these methods are in fact fully geometrical in the formal manner; but they do rely on certain mechanical conceptions, like the generation of curves by moving lines or by the intersection of solids of revolution.[31] The student of Plato's abstract philosophy might well view the latter with disapproval. On the other hand, Eratosthenes' own solution entailed the production of a mechanical device, and in his eyes its practical viability is a major asset; in this context, the practical limitations of the earlier designs could well be a point of note, inducing him to underscore their purely theoretical nature. Here again, the discrepancy is undoubtedly due to Eratosthenes himself. The intent in the letter is historical, but that in the *Platonicus* is dramatic and philosophical. He might thus take certain liberties in the latter, denied to him in the former. To set the scene for his dialectical point, he can elaborate the mechanical elements more explicitly than the original treatments actually did. Such fictionalization would surely be acceptable in a work which made no pretension of being plainly historical.

Accepted as a serious historical source by a man well placed for a knowledge of this period of Greek mathematical history, Eratosthenes' text as reported by Eutocius offers hope of insight into matters often obscured in modern discussions. First, although historians typically wish to leave open the question of the historical validity of the Delian oracle,[32] one now readily perceives the story to be a fabrication, originating from within the Platonic Academy around the middle of the 4th century B.C. By this time the problem of cube duplication was already familiar, for as Eratosthenes himself notes, important advances in its analysis were made by Hippocrates of Chios, that is, almost a half-century before.[33] It would seem odd indeed for the Delian oracle to be concerned with an old problem which then happened to be eluding the efforts of contemporary geometers. But on the other hand, as a dramatic way of affording recognition and motivation to those efforts, the story makes good sense, especially since geometers associated with the Academy were prominent in this activity. But one can only guess what the context and purpose of the story might have been at its first composition. As we have indicated, the specific morals attached to it by Eratosthenes (in the *Platonicus*) and by Plutarch seem to be their own additions, but they are likely to have sensed accurately the ancient Academicians' view of these matters. Any student of the *Republic* can appreciate the strength of Plato's insistence on purity and abstractness of mathematical entities.[34] The kinematic element in the constructions for the cube duplication might thus provoke discussion of the relative

status of geometry and mechanics, and indeed this very issue is important within Aristotle's theory of the order of the sciences.[35]

Furthermore, Eutocius' text of Eratosthenes is noteworthy for the sparseness of the historical information it transmits. One should have thought that Eratosthenes would be concerned with producing as full an account as he could of the studies preceding his own, and his text does leave that impression. Yet it is amazing that he turned up with so little. He appears to have discovered nothing worth mentioning from the whole century or so separating him from the efforts of the Academicians. As for the even earlier efforts, he cites only the work of Hippocrates and a scene from an unnamed tragic poet.[36] In the latter, when King Minos is told that the tomb he had ordered for Glaucus will have dimensions of one hundred feet on a side, the king answers that this would be too small, and so, "let it be double; without mistaking its fine form, swiftly double each member of the tomb." The new structure will of course be eight times larger than the original one in volume, not its double. Eratosthenes calls attention to the poet's error. But actually there is none here: the poet intends that each dimension shall be doubled; he is not articulating the mathematicians' problem of the cube duplication. Evidently, Eratosthenes has misconstrued the passage in his desire to find precedents for an interest in this problem.

In effect, Eratosthenes has sought some motivation for Hippocrates' study of the cube duplication. But Hippocrates' effort by its very nature reveals that motive; for in Eratosthenes' account, this is what he does:

> It used to be sought by geometers how to double the given solid while maintaining its shape. . . After they had all puzzled for a long time, Hippocrates of Chios was first to come up with the idea that if one could take two mean proportionals in continued proportion between two lines, of which the greater is double the smaller, then the cube will be doubled. Thus he turned one puzzle into another one, no less of a puzzle.[37]

Hippocrates' insight is of course not restricted to lines assumed in a 2 : 1 ratio. If for any two given lines, A and B, we can insert the two mean proportionals, X and Y, then $A : X = X : Y = Y : B$. Thus, by compounding the ratios, one has $(A : X)^3 = (A : X)\,(X : Y)\,(Y : B)$, that is, $A^3 : X^3 = A : B$. Thus, X will be the side of a cube in the given ratio (B : A) to the given cube (A^3).[38]

In his closing line here, Eratosthenes seems to perceive little merit in Hippocrates' move. But in fact, by this stroke Hippocrates has put the problem into a form permitting the application of a whole new range of geometric techniques, those of proportion theory. In general, this procedure of "reducing" (*apagōgē*) one problem to another from whose solution its own would follow is a powerful technique of problem solving, a forerunner of the fruitful method of geometric "analysis." As Proclus tells us, Hippocrates was the first geometer known to have applied the method of reduction in the investigation of "puzzling diagrams," i.e. difficult problems of construction.[39] We will see later that Hippocrates appears to have adopted a similar strategy in the investigation of the circle quadrature.

Later geometers clearly recognized that such a *reduction* does not in itself amount to a *solution* of the problem posed. But would Hipppocrates already have made this distinction in the case of his treatment of the cube duplication? A passage from Aristotle suggests that he did. In *de Anima* II, Ch. 2, Aristotle observes that one may define a term like "squaring" (*tetragōnismos*) not only by saying *what* it is (i.e., the equality of a square with a rectangle), but also by stating its *cause* (i.e., the finding of a mean proportional).[40] Now, the latter refers to the reduction of the given problem of areas, such as one may read in Euclid's VI, 17, and is the precise plane analogue to Hippocrates' problem of solids. But the construction of this planar problem, whether in the form of a squaring (as in *Elements* II, 14) or in that of finding a mean (as in VI, 13), was certainly already available to Hippocrates. A long fragment of his study of the quadrature of crescent-shaped figures survives, and the techniques of proportions are applied throughout to situations far more demanding than the squaring of rectangles.[41] Since the distinction between the reduction and the solution was clear in the instance of the planar case, the same must surely have been appreciated in the solid case as well. We may suppose, then, that Hippocrates and his followers continued their research into the cube duplication by seeking the construction of the two mean proportionals, where the latter effort achieved its first clear successes through the discoveries of Archytas, Eudoxus, and Menaechmus.

These considerations help clarify the way in which the cube duplication was first articulated as a problem. Within the study of the measures of plane figures, affected via the techniques of proportions, it would be perfectly natural for a geometer like Hippocrates to consider the case of two mean proportionals and so perceive its relation to the problem of volumes, by analogy with the case of the single mean. In this way, the new problem emerges through the natural development of geometric research. Now, it sometimes happens that research moves in new directions in response to external stimuli: practical problems arising from commerce, government, and engineering are a manifest basis of much in the most ancient mathematical traditions; ritual might sometimes play a role; or philosophy might instil a sensitivity to formal questions.[42] The story of the Delian oracle has thus often been used as an example of such external motivation. But one can now see that the ancient evidence on the origins of the cube duplication affirms quite a different view: that the legend arose within Plato's Academy around the middle of the 4th century B.C., long after the problem had attained notoriety through the work of Hippocrates; and that Hippocrates himself articulated this problem as a natural extension of the geometric field of his time. In the absence of direct testimony to earlier work in the 5th century, one has no grounds for supposing a development of any other sort.

THE QUADRATURE OF THE CIRCLE

In connection with the problem of constructing in a given angle a parallelogram equal to a given rectilinear figure (*Elements* I, 45), Proclus remarks:

> Having taken their lead from this problem, I believe, the ancients also sought the quadrature of the circle. For if a parallelogram is found equal to any rectilinear figure, it is worthy of investigation whether one can prove that rectilinear figures are equal to figures bound by circular arcs.[43]

Although the words "I believe" reveal this view to be merely a surmise on Proclus' part, rather than a conclusion based on more or less explicit testimony; it is far from unreasonable. Indeed, there appears to be a certain sense of the problem of circle quadrature in remnants of the ancient Egyptian and Mesopotamian traditions as far back as the middle 2nd millennium B.C. and earlier. Specifically, the scribe of the Rhind Papyrus works out the area of a given circle first by subtracting 1/9 of its diameter and then squaring the result.[44] Thus, in effect, the circle is equated with the square whose side equals 8/9 the diameter of the circle. This rule is illustrated by a figure in which a square is divided into nine equal squares, where the diagonals of the four small squares at the corners are drawn in. In this way one sees at once that the circle might be approximated as 7/9 the enclosing square. One might conjecture that the scribe went on to subdivide each of the squares into nine squares and then tried to estimate how many of the 81 resulting squares combined to approximate the circle. But another approach might have been followed. Knowing that problems of areas often require computing squares and their roots, the scribe might have sought a value near 7/9 which resulted from a squaring. Since $7/9 = 63/81$, where the latter is just short of $64/81 = (8/9)^2$, a quasi-deductive route might be possible for deriving the rule applied by the scribe of the papyrus.

If such was indeed the basis of the rule, the scribe would also have a basis for recognizing the measurement of the circle as a geometric problem. The same would not be true within a tradition relying entirely upon empirical derivations of its geometric rules. It is hardly possible, for instance, that the Egyptians could have known the 1 : 3 ratio of the volumes of the cone and cylinder having equal altitude and base other than by observing that the fluid contents of a conical vessel can be emptied exactly three times into a cylindrical one, or by some comparable physical measure.[45] But inaccuracies of manufacture, spillage, and other such factors would render this procedure approximate. There would thus be no firm demarcation between rules which are exact (as is that for the ratio of volumes) and those which are approximate (e.g., the circle measurement). Certainly, within the early traditions the aim was to obtain practical procedures. The project of somehow producing a construction of the square equal to the circle seems remote from this objective, even if it may be perceived as entailed within the results given. Its explicit formulation as a problem must then be sought elsewhere.

We may gather that Greek mathematicians had hit upon this formulation of the problem before the close of the 5th century B.C. For in his comedy *The Birds* (staged in 414 B.C.) Aristophanes presents this lampoon of the astronomer Meton arriving as self-appointed civic surveyor of Cloud-cuckoo-land:

> If I lay out this curved ruler from above and insert a compass—do you see?— . . . by laying out I shall measure with a straight ruler, so that the circle becomes square for you.[46]

Now, there is no report of Meton's inquiring into the circle quadrature, and the passage here actually views him as somehow dividing the circle into quadrants.[47] But the force of the jest would seem to require that the audience perceive it to refer to a real problem under investigation by geometers at that time. This conforms with reports we have of certain efforts by Hippocrates, Antiphon, and Anaxagoras.

Relative to Hippocrates, Aristotle tells that he proposed a false proof of the quadrature of the circle "by means of segments" or "by means of lunules."[48] The latter refers to Hippocrates' masterful construction and quadrature of crescent-shaped figures bounded by arcs of circles; we turn later to the question of the bearing this effort had on the circle quadrature. According to the account preserved by Simplicius, the 6th-century Aristotelian commentator, drawing explicitly from a report by Eudemus, Hippocrates prefaced his treatment with demonstrations of certain basic theorems on circles:

> He made his start and set down first some things useful for those [constructions], that segments of circles have to each other the same ratio as their bases in power; and this he proved from having proved that the diameters have the same ratio in power as the circles. . . .[49]

This passage notes explicitly that Hippocrates *proved* these theorems. Moreover, it goes on to supply further details of the derivation of the segment theorem in good agreement with a version given much later by Pappus.[50] We may thus be confident that toward the close of the 5th century, Hippocrates already appreciated the essential concerns of the deductive form entailed by the proofs of theorems such as these. Now, the lines just cited include the statement of a theorem identical to one in Euclid's *Elements* (XII, 2): "Circles are to each other as the squares on their diameters." Simplicius' terminology of "powers" (*dynameis*) instead of "squares" (*tetragōna*) reassures us that here he has followed the archaic Eudemean usage in his text, rather than improvise through his own knowledge of Euclid.[51] But we encounter difficulty in supposing that Hippocrates advanced the proof of this theorem on circles in its Euclidean form. For the limiting method on which that depends is due to Eudoxus, as we learn on the good authority of Archimedes.[52] Then what sort of proof could Hippocrates have provided a half-century or more before Eudoxus?

Aristotle's remarks on the false efforts at circle quadrature link Hippocrates

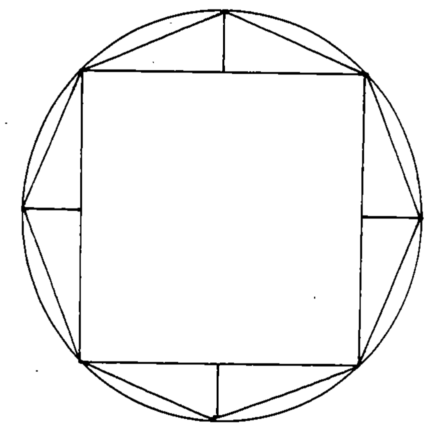


Figure 1

with two sophists from early in the 4th century, Bryson and Antiphon.[53] What Simplicius reports of the latter may help us here. Antiphon proposed to construct a rectilinear figure equal to a given circle by a procedure involving the successive doubling of the number of sides of an initial inscribed polygon. For instance, if one starts with the inscribed square, erects the perpendicular bisector to each side, marks the point where each meets the circle, and connects each of these points to the adjacent vertices of the square, one will have erected an isosceles triangle on each side of the square, and their combination with it will produce the inscribed regular octagon. (See Figure 1.) One may next bisect the sides of this octagon, ultimately resulting in the inscribed 16-gon; and so on.

> Continuing in this way, at the point where the area was exhausted, he said he will have inscribed a certain polygon in the circle in this manner, such that its sides conform to the arc of the circle on account of their smallness. Since we are able to produce the square equal to any polygon, . . . because the polygon has been produced conforming to the circle, we shall also have produced a square equal to the circle.[54]

Antiphon's step yielding an arc conforming (*epharmozein*) to the sides of the inscribed polygon would appear to be drawn from notions familiar through the ancient atomists. For them, the microscopic world of atomic reality might be qualitatively different from the macroscopic world as perceived by our senses. Specifically, the circle appears to meet its tangent at a single point only, and we insist on this as part of the abstract conception of this configuration; but in their physical manifestation they will meet along a small line segment. This very question was discussed by Democritus and some of the sophists, notably Protagoras; and one of Simplicius' sources on Antiphon, the 3rd-century-A.D. commentator Alexander of Aphrodisias, observes that this principle—the circle's meeting the tangent line at one point only—was the one whose abolition lay at the root of Antiphon's fallacious quadrature.[55] We thus perceive a firm dialectical context for Antiphon's argument.

The invalidity of his circle quadrature is evident, for it relies upon the actual "exhaustion" (*dapanan*) of the whole area lying between the circle and the polygons, while the indicated procedure of inscribing polygons will not do this if applied only a finite number of times.[56] Hence, for Aristotle and the commentators, its refutation is not a matter for the geometer to be concerned about. By contrast, modern scholars have tended to assign to Antiphon credit for the introduction of the important notion of polygonal approximation, essential for the ancient quadratures of curvilinear figures.[57] But is this a plausible position to maintain? Given that Antiphon and Hippocrates were near contemporaries, the latter without doubt being rather the older, shall we suppose that the sophist, in the course of framing a muddled geometric argument to support his dialectical position, lent essential insights to the geometer? Or is it not more likely that he was drawing from and modifying procedures already familiar among geometers? The latter certainly typifies the later philosophical tradition, as represented by Plato, Aristotle, the Stoics, and the Sceptics, for instance, and their stance toward the technical sciences of their times. By contrast, it seems unclear why a geometer like Hippocrates should turn to the sophists for technical assistance in a theorem of this sort. The fact that Antiphon's use of this technical procedure lapses into patent fallacy, while Hippocrates requires the same procedure for the proof of a theorem implemented with full logical precision in his quadratures of lunules, merely confirms one's doubts about Antiphon's originality in this technical context.

If we assign to Hippocrates knowledge of the polygonal procedure preserved in our text of Antiphon's quadrature, then we obtain a good basis for reconstructing a proof he could have used for the theorem on the ratios of circles. For one notes a subtle difference between Antiphon's method and the one used in the Euclidean proof of this theorem (XII, 2). Euclid obtains each polygon in the sequence by bisecting each of the arcs cut off by the vertices of the polygon preceding it. Antiphon, on the other hand, builds each new polygon from its precursor by adding the isosceles triangles onto each of its sides. Now, if one considers two circles, each enclosing similar polygons constructed in this way, since the initial polygons and the added triangles taken pairwise are similar, these figures will be in the same ratio, namely that of the squares of the diameters of the circles. Hence, the *sum* of *all* these figures, extended indefinitely, will be in the same ratio. Since these sums constitute the circles themselves, it follows that they too have the ratio of the squares of their diameters, as Hippocrates asserts.[58]

From the strict logical viewpoint, of course, this argument fails, for it applies to the infinite case a theorem on the ratios of sums which can be established for sums of only a finite number of terms, if even that, given the technique of proportions available to Hippocrates.[59] But this approach is so strongly founded on clear notions of the continuity of magnitude that the difficulty might not necessarily have been appreciated at an early stage of these studies. We must assign to Eudoxus the first full awareness of this difficulty and the discovery of

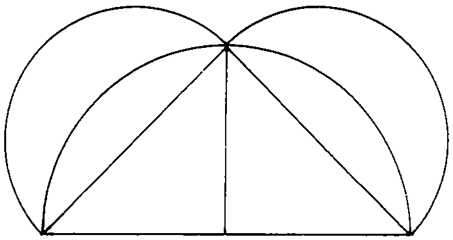
the method of limits by which it could be removed. It is noteworthy that the technique of sums we have just proposed is a standby within Archimedes' demonstrations; that Archimedes provides a proof, but only for the finite case (*Conoids and Spheroids*, Prop. 1); nevertheless, that he applies its infinite case in all of the planar and solid measurements presented in his *Method*.[60]

One cannot say for certain whether such studies of the circle quadrature can be set much before the time of Hippocrates. Writing in the 2nd century A.D., Plutarch mentions that the 5th-century natural philosopher Anaxagoras "drew the quadrature of the circle" during a spell in prison.[61] This would place an awareness of the theoretical character of the problem of circle quadrature to not much later than the middle of the 5th century B.C. It is an especially intriguing attribution, in that Anaxagoras insisted on the continuity of matter as the basis of the nature of things and seemed to perceive some of the mathematical implications of the principle of indefinite divisibility, the very principle later violated in Antiphon's circle quadrature. To be sure, Proclus reports that Anaxagoras "touched on many things relating to geometry."[62] But the ancient tradition otherwise assigns to him no specific interest in mathematics, and Plutarch's testimony is quite casual, inserted merely to illustrate the point that the thinking man finds happiness even under trying circumstances. The subsequent notoriety of the circle quadrature might all too easily color the testimony transmitted by later authorities. Thus, we must forswear using Anaxagoras' alleged contribution as a sign of early interest in the problem.

From Aristotle we may gather that a construction of the circle quadrature was attempted on the basis of Hippocrates' quadrature of lunules. For in his account of the logical method of "reduction" (*apagōgē*) he remarks thus:

> We are the closer to knowing, the fewer the mean terms in the syllogism. For instance, as the circle together with the lunules is equal to a rectilinear figure, we are that much the closer to the squaring of the circle itself.[63]

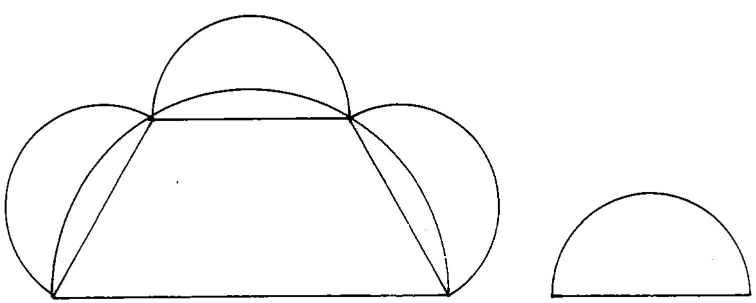
Viewed in the light of Hippocrates' reduction of the cube duplication to the problem of the two mean proportionals, this passage seems to indicate that Hippocrates attempted a similar strategy for the circle quadrature: to cast it into an alternative form permitting the application of a wider range of the techniques then known. This is fully borne out in the extended discussion of circle quadrature presented by Simplicius commenting on a related passage from the *Physics* where Aristotle mentions the false quadratures.[64] In addition to sketching and criticizing Antiphon's argument on the circle quadrature, as we have already seen, and noting several other efforts on this problem, Simplicius gives a detailed account of Hippocrates' quadrature of lunules. Indeed, he provides two different versions, a simplified treatment known through Alexander's commentaries, followed by an elaborate series of constructions and proofs quoted verbatim from Eudemus' history. This passage from Eudemus is our most important fragment from the pre-Euclidean geometry, providing invaluable insight into the content, methods, and terminology of the early tradition. We shall examine it in detail, after a brief consideration of the shorter version from Alexander.



(a) **Figure 2**

In Alexander's text, two cases of lunules are presented[65]:(a) Starting from a right isosceles triangle, we draw on each of its sides a semicircle. (See Fig. 2a.) Since the square on the hypotenuse equals the sum of the squares on the two legs, and since circles are as the squares on their diameters, it follows that the semicircle on the hypotenuse equals the two semicircles on the legs. If we now remove the portion common to the large semicircle and the two smaller ones, we find that the isosceles triangle which remains is equal to the two lunules which arch over its legs. Thus, by equating the lunules to a rectilinear figure, we have effected their quadrature.[66] (b) In the second case, we begin with the trapezium formed as a bisected regular hexagon and draw semicircles on each of its sides. Since the longest side of the trapezium is double each of its shorter sides, the large semicircle equals four of the smaller ones. As before, subtracting the common portion, we find that "the remaining lunules together with the (small) semicircle are equal to the trapezium." (See Fig. 2b.)

Recalling Aristotle's remark on the reduction of the quadrature of the circle to that of the lunules, cited above, we see that he evidently had this construction of Hippocrates in mind. But precisely what conclusion Hippocrates himself intended to draw from it has been debated among ancient and modern com-



(b) **Figure 2**

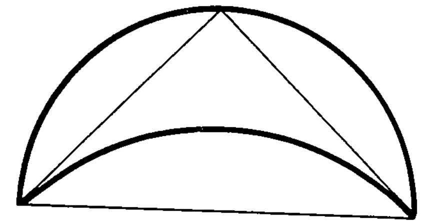


Figure 3

mentators alike.[67] Alexander, for instance, supposed that Hippocrates wished to argue that since the lunule in (a) has been squared, while the sum of the circle and the lunule in (b) has also been squared, then the circle itself has been squared. If so, he would have committed the patent error of assuming that the quadrature of a single case of lunule amounts to the quadrature of all cases. Simplicius, by contrast, found it incredible to conceive that a geometer of Hippocrates' outstanding caliber could have lapsed into such a blunder. Under this view, one would charge to others, for instance, to Aristotle, the attempt to construe Hippocrates' results as a circle quadrature. Doubtless, this is the correct view, for as we saw in the parallel instance of the cube duplication, Hippocrates' reduction of the problem there was not likely to have been regarded as a solution of it. As we shall see later, Simplicius' text provides further insights into Hippocrates' view of the circle quadrature.

The alternative account, which Simplicius quotes "word for word" (*kata lexin*) from Eudemus, presents the full treatment of four cases of lunules. The first is essentially identical to Alexander's case (a), while the fourth bears comparison, if somewhat more loosely, with his case (b). The other two cases are new, constructing lunules which are in the one instance greater and in the other less than the arc of a semicircle. Translations and technical summaries of this text are readily available, so that a brief summary will suffice here.[68] My principal aim now is to retrieve the line of thought underlying Hippocrates' constructions and thus to reveal his sense of the more general class of lunules.

(i) In the first case, Hippocrates begins with the right isosceles triangle and circumscribes the semicircle on its hypotenuse. (See Fig. 3.) This much conforms with case (a). But now, instead of drawing semicircles on the legs also, he draws on the hypotenuse the circular segment similar to those which appear on the legs of the triangle. In this way he obtains a single lunule, the figure bounded by the two circular arcs, and shows that it equals the initial triangle. This follows, since the similar segments are as the squares on the chords which define them, so that the larger segment equals the sum of the two smaller ones. If to the mixtilinear figure inside we add the two small segments, we obtain the lunule; if we add to it, alternatively, the large segment, we obtain the triangle. Thus, the triangle and the lunule are equal to each other. It is clear that this lunule is the same as each of the two lunules constructed in case (a). But the present treatment reveals a greater technical range. For instance, its author operates freely with similar

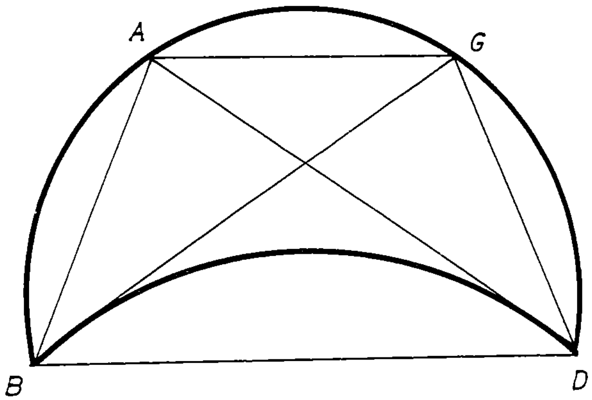


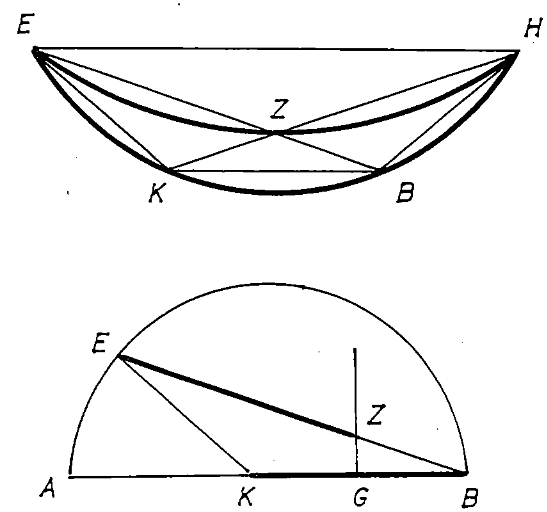
Figure 4

segments, not just semicircles. The pertinent definition, given earlier in the text, stipulates them as subtending equal angles in their respective circles. As we saw above, his proof that similar segments, like similar rectilinear figures, have the ratio of the squares on homologous sides signifies that major progress has already been made toward the Eudoxean manner of measuring curvilinear figures. In all these respects, the treatment of case (i), despite its relative simplicity, prepares us nicely for the more intricate constructions which follow.

(ii) Hippocrates now assumes the construction of a trapezium whose base has to each of its three remaining equal sides the ratio $\sqrt{3} : 1$. (See Fig. 4.) He next draws the circumscribing circle and then draws on the base the circular segment similar to those which have thus been formed on the three other sides. The text establishes in full detail that the arc of the outer bounding circle is greater than a semicircle.[69] The actual proof of the quadrature of the lunule is omitted from Eudemus' text, so Simplicius supplies his own, remarking that Eudemus must have considered it obvious. But in fact the analogous proofs are given in full for each of the following two cases, and textual considerations show these to be in the pre-Euclidean manner of Eudemus.[70] Thus, one may infer that Simplicius' text came to lose portions of the proof through scribal omissions.[71] At any rate, since the one large segment equals the three small ones, the same process of addition as used in (i) shows that the lunule equals the trapezium.

From these two cases we may recognize the condition Hippocrates has in mind for the construction of squarable crescents. If we conceive of a rectilinear figure bounded by two polygonal arcs, the outer consisting of n sides each equal to s_n, the inner consisting of m sides each equal to s_m, and if $s_n{}^2 : s_m{}^2 = m : n$, and if furthermore the sides s_n, s_m subtend equal angles in the circles circumscribed about their respective polygonal arcs, then the lunule bounded by these circular arcs will equal the rectilinear figure bounded by the polygonal arcs. Although Hippocrates does not formulate the construction in such general terms,

(a)



(b)

Figure 5

his treatment of the third case indicates that a conception of this sort did indeed guide his thought. His examination of this next case, like those which preceded it, is entirely in the synthetic mode, and so tends to obscure the manner of its discovery. I shall thus attempt to reconstruct on its basis an alternative version in the analytic mode in order to display its essential idea.[72]

Let an investigation of the lunule of the type just stated be proposed, where the two bounding arcs correspond to polygonal arcs, the one of three equal sides, the other of two. Thus, $s_2 : s_3 = \sqrt{3} : \sqrt{2}$. Let us conceive that the figure has been produced, consisting of the double arc EZH and the triple arc EKBH (see Fig. 5a); if we pass a circle through each of these polygonal arcs, then we must also have that the circular arcs EZ, ZH, EK, KB, BH all subtend the same angle at the center of their respective circles. We join EH and consider the angles ZEH and KEH. As the former intercepts the single arc ZH, while the latter intercepts the double arc KBH, it follows that angle KEH is double ZEH, so that EZ bisects angle KEH and its extension must pass through B. Furthermore, point Z lies on the perpendicular bisector of BK[73], while point E lies on the circle of radius EK and center K. Thus, given lengths in the ratio $\sqrt{3} : \sqrt{2}$, to accomplish the construction we need only take the shorter length as KB, draw the circle having the same length as radius and center at K, draw the perpendicular bisector of KB, and then so place a line passing through B that the portion of it intercepted between the circle and the perpendicular bisector equals the larger of the given lengths (see Fig. 5b). In this way, both the placement and the length of EZ will be determined, and from it the rest of the figure can be completed.

The latter sets out precisely the same form in which Hippocrates accomplishes the construction of case (iii) of the lunules. In particular, one perceives through this analysis how natural it is to use the placement of EZ for effecting this. In Hippocrates' terms, "the line *inclining* (*neuousa*) toward B and intercepted between the line DG and the arc AEB shall have the given length."[74] This is the earliest known instance of the use of the constructing technique called *neusis* by the Greeks. As we shall later see, it plays a prominent role in many of the problem-solving efforts by Archimedes and his followers a century and a half after Hippocrates.[75]

Having so constructed the lunule, Hippocrates next proves that it is equal to the associated rectilinear figure EZHBK; for since $3\ EK^2 = 2\ EZ^2$, the three outer segments will equal the two inner ones. Since, furthermore, "the *mēniskos* (lunule) consists of the three segments together with that part of the rectilinear figure apart from the two segments,"[76] the lunule and the rectilinear figure are equal. Hippocrates goes on to show that the outer bounding arc of this lunule is less than a semicircle. At this point Eudemus injects an amazing remark:

> In this manner Hippocrates squared every lunule, since in fact (*eiper*) [he squared] the [lunule] of a semicircle and the one having the outer arc greater than a semicircle and the one [having the outer arc] less than a semicircle.... But he squared a lunule together with a circle as follows.[77]

What are we to make of "*every* lunule"? Does it signify merely the three particular cases just examined? But in context a more general sense seems to be indicated, since one moves at once to the consideration of the case pertaining to the circle quadrature. Then has Eudemus erred in supposing that squaring one lunule or even a few is tantamount to squaring them all? If this is so, then did Hippocrates himself make the same error, thus bringing on himself Aristotle's charges of having produced a fallacious argument on the circle quadrature? But it is hard to imagine how the one responsible for these carefully ordered proofs on the lunules could then have committed such an elementary blunder in reasoning. Now, there is indeed a sense in which Hippocrates might claim to have established the quadratures of a whole class of lunules, namely those for which the m similar segments along one bounding arc equal the n similar segments along the other arc. Although, of course, he has not produced the general *construction* of these figures, he can assume that their *quadratures* are obvious by virtue of the three cases given. If this is what Hippocrates intended by asserting that "every lunule" had been squared, we would infer that Eudemus, and doubtless others before him, had misconstrued his claim as referring to every possible form of lunule.

As for case (iii), so also for (iv), the synthetic mode adopted by Hippocrates conceals its essential line of thought. But it emerges if one views this case in relation to case (b) in Alexander's version, by analogy with the relation of cases (i) and (a). If we start with the hexagon inscribed in a circle, instead of using either its side or its diameter as base, let us introduce the chord connecting the two nonadjacent vertices H and I. (See Fig. 6.) Thus, HI is the side of the

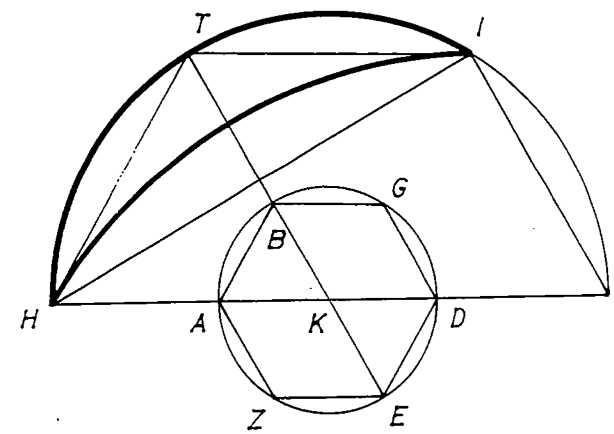


Figure 6

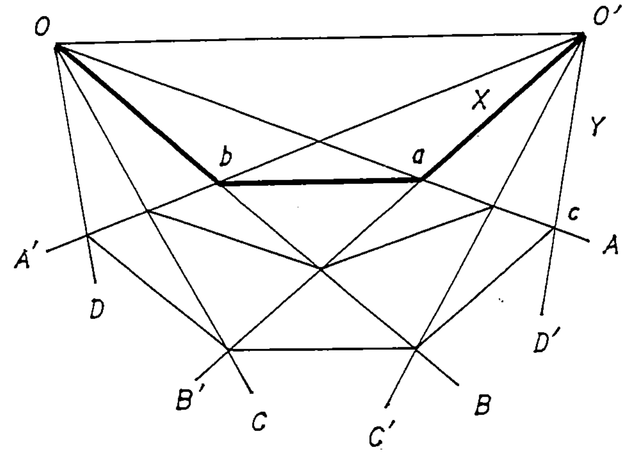
inscribed equilateral triangle, while HT is the side of the hexagon, so that $HI : HT = \sqrt{3} : 1$. If we draw over HI a segment similar to each of those over HT, TI, it will equal three times either of those segments. In the familiar manner, it follows that the triangle HTI equals the lunule together with one of those smaller segments. Thus, squaring the lunule reduces to squaring that segment. Hippocrates conceives of a set of segments similar to it and equal to it in sum; these are the segments on the sides of a hexagon inscribed in a circle whose diameter has to that of the initial circle the ratio $1 : \sqrt{6}$. The segment over HT thus equals the sum of all six segments on the sides of the hexagon ABGDEZ. Adding to both the area of the hexagon and then combining with the earlier result, we find that the triangle together with the hexagon is equal to the lunule plus the smaller circle. In Eudemus' text the construction begins with the drawing of the two circles, so that the motivation behind the choice of $1 : \sqrt{6}$ as the ratio of their diameters remains unclear until well into the proof. Having established the equality just stated above, he concludes:

> Thus, if the rectilinear figures [i.e., the triangle and the hexagon] mentioned can be squared, so also can the circle with the lunule.[78]

This ends the quotation from Eudemus; for Simplicius observes in the very next sentence:

> One ought to place greater trust in Eudemus [than in Alexander] to know things pertaining to Hippocrates of Chios, for he was nearer to his times, being a disciple of Aristotle.

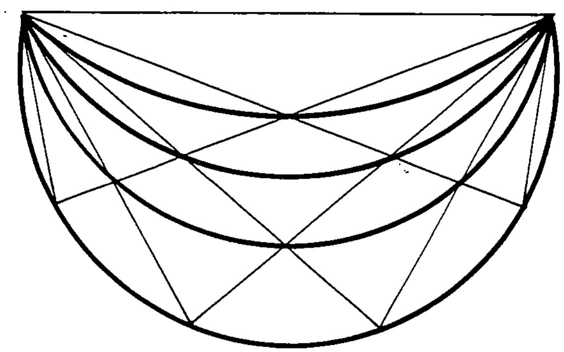
It is clear in Eudemus' account that Hippocrates makes no presumption of having squared the circle itself, but only the circle in combination with the lunule. This



(a) **Figure 7**

is so clear, in fact, that Simplicius questions the view that Hippocrates intended this here at all, and so suggests that when Aristotle speaks of his fallacious quadrature of the circle "via segments," he must refer to an entirely different argument.[79] Far more likely, however, is that Hippocrates presented this last case of lunule as a successful *reduction* of the circle quadrature, without its being a *resolution* of that problem. He would thus leave for later research the investigation of this lunule. If some followers rashly asserted that the circle had indeed been squared in this way, we have no grounds at all for charging this error to Hippocrates himself or even for suggesting that he wished slyly to leave this impression.

Aside from Hippocrates' attitude toward the circle quadrature, we should consider what his view of constructions was. It will be helpful to consider first the more general class of lunules and certain modern contributions to their study. At each end of the given line segment 00' let there be drawn a system of equally spaced rays, 0A, 0B, 0C, 0D, . . . , and 0'A', 0'B', 0'C', 0'D', . . . , respectively, where in each case the angle θ separating consecutive rays is the same. (See Fig. 7a.) Then we may form a polygonal arc from 0 to 0' by selecting the intersections of appropriate rays from each system. For instance, if point a is the intersection of rays 0A, 0'B' and point b is that of 0B, 0'A', then the broken line 0'ab0 will be a polygonal arc consisting of three equal lengths; for if we pass through them the circular arc joining 0 and 0', each length will be a chord subtending angles equal to θ relative to the points 0 and 0'. Thus, the circular segments having these chords as base will be similar to each other. (See Fig. 7b.)[80] In analogous fashion, we can construct between 0, 0' another polygonal arc having as many equal elements as desired.[81] As before, we obtain a set of circular segments similar to each other and to the segments associated with any other such polygonal arc constructed via the same initial sets of rays. Let two such arcs be taken, the inner having n elements, the outer having m elements. Denoting by X a side of the inner arc (e.g., 0'a) and by Y a side of the outer



(b) **Figure 7**

arc (e.g., 0'c), and applying the identity of sines to the triangles 00'a and 00'c, we have $X : Y = \sin m\theta : \sin n\theta$. The condition that the lunule bounded by the two corresponding circular arcs be squarable in Hippocrates' sense is that $nX^2 = mY^2$, so that $\sin m\theta : \sin n\theta = \sqrt{m} : \sqrt{n}$. If the general case is denoted by the pair (n, m), then the three cases examined by Hippocrates are (1, 2), (1, 3), and (2, 3). These all happen to be constructible in the familiar "Euclidean" sense, that is, by means of compass and straightedge alone. That two other cases, (1, 5) and (3, 5), are of the same sort was noted by Wallenius (1766) and Clausen (1840)[82]; for one can readily see that their defining relations reduce to a quadratic in $\cos 2\theta$. Furthermore, researchers have established that the lunule $(1, p)$ is *not* constructible in this manner when p is a prime not of the Gaussian type (that is, of form $2^{2^k} + 1$) (Landau, 1903), nor is the lunule (n, p) so constructible for $n = 1, 2, \ldots, p - 1$ (Tschakaloff, 1929).[83] Moreover, even for Gaussian prime p (other than 2, 3, and 5), the lunule $(1, p)$ is not constructible (Wegner); furthermore, apart from the cases already known to be constructible, the lunule (n, m) is not constructible when n, m are both odd integers (Tschebotaröw, 1935).[84]

For our purposes now, it is important to realize that this modern conception of the problem of constructing the lunules is anachronistic as far as Hippocrates' study is concerned. For we recall that his construction of lunule (2, 3) depended on a *neusis*, that is, the suitable inclination of a line segment of given length. As it happens, an alternative mode for effecting the *neusis* he requires is possible via planar methods (i.e., compass and straightedge), for it may be expressed as a quadratic relation falling within the class of problems known as the "hyperbolic application of areas," so that a solution can be worked out on the model of Euclid's *Data*, Prop. 59, based on *Elements* II, 6 and VI, 29.[85] While it is a reasonable conjecture that Hippocrates had access to these techniques, nothing in our text directly supports that claim. Certainly, his manner of introducing the *neusis* gives no hint of requiring us to effect this step by any means other than the obvious: the manipulation of the marked line EZB.[86] Indeed, even later geometers like Archimedes freely admit *neuses* in their constructions, where sometimes, as in *Spiral Lines*, Prop. 5, an alternative "Euclidean" method is possible.[87] Thus, I would maintain that Hippocrates' goal was to find any construction he could. If he proceeded from the general conception of the configuration given above, he might perceive that the construction of any lunule (n, m) satisfying the condition $X : Y = \sqrt{m} : \sqrt{n}$ reduces to a determination of the appropriate angle θ. Initially (for $\theta = 0$) the ratio $X : Y$ equals $m : n$. By experience of actually constructing figures, he could realize, if not prove, that as θ increases the ratio decreases, and hence that the specific configuration sought is possible.[88] In principle, his admission of *neuses* as a constructing technique makes accessible to him a wider range of cases than just those of the "Euclidean" type, including, for example, cases reducing to cubic relations.[89] Thus, the fact that he presents only the three squarable cases indicates that neither he nor any other ancient geometer who tried his hand at this problem was able to come up with any cases besides these. In this instance, then, not having recourse to the later algebraic methods in geometry constituted a real barrier against the discovery of solutions to these problems.

As noted above, Simplicius assigns greater weight to Eudemus' text than to the one transmitted by Alexander, believing that the former more closely represents the work of Hippocrates himself. Now, from the purely *textual* point of view, he seems quite correct in maintaining this. In certain contexts, for instance, Eudemus adheres to older terminology discontinued in the tradition after Euclid: he speaks of "powers" (*dynameis*) instead of "squares" (*tetragōna*), and uses expressions like "the line *on which (eph' hēi)* AB" about as frequently as their short forms (e.g., "the line AB," or simply "the AB"), while only the latter are found in the later formal tradition.[90] By contrast, Alexander's text is cast in full conformity with Euclidean usage. On the other hand, from the *technical* viewpoint, Alexander's constructions of lunules (a) and (b) are rather simpler than the analogous cases (i) and (iv) of Eudemus; and the technical refinement evident throughout the proofs of Eudemus' four cases clearly surpasses the treatments in Alexander. It is of course conceivable that Alexander's text originated from an effort to simplify an earlier version in the manner of Eudemus. But we recall that in his passage on "reduction," Aristotle alludes to a quadrature set in the form of Alexander's case (b), rather than Eudemus' case (iv). This affirms that Alexander's version too derives ultimately from a 4th-century source. Its simpler treatment of the lunules well suits what one would expect of the early phases of such a study; and as we have seen, it assists considerably in retrieving the motivations underlying the more elaborate constructions given by Eudemus.

This raises the question of the provenance of Eudemus' text. Much depends on Eudemus' reliability and sophistication as a historian. While we possess far too little of his work to make a secure judgment on this matter, it would seem that the purposes occasioning his account of Hippocrates would be admirably served through a fair transcript of his source materials, while Eudemus' motives for producing a radically modified account, indeed even his competence to do so in such a detailed technical context, would be entirely unclear. Can we then assume that Eudemus' text reflects with reasonable accuracy an actual writing of Hippocrates? There is of course no difficulty in supposing that Hippocrates' initial version of the study of lunules was in the simpler form indicated by Alexander, and that he later produced the more refined version recorded by Eudemus. On the other hand, it is equally plausible that the latter version grew out of the researches of followers inspired by his efforts to pursue this study. Doubtless, Eudemus was in the position to distinguish between a genuine writing of Hippocrates and a later writing in the tradition instigated by him. But would he insist on making this distinction clear in his discussion of the quadratures? We can hope that he would, yet must admit that it is possible he did not.[91]

These considerations affect one's use of the fragment as an index of the form and terminology of 4th-century mathematical writing, for we cannot with certainty locate it closer to the time of Hippocrates, toward the beginning of that century, than to the time of Eudemus, nearer its end. Nevertheless, its reliability as a testimony of Hippocrates' findings seems secure, for we have Eudemus' explicit statement several times in the fragment that the four cases of lunules were all studied by him, and it is hard to suppose that he could have been mistaken in this regard. Furthermore, the proofs of the quadratures, even in the simpler version reported by Alexander, indicate advances in the geometric tradition around Hippocrates' time; access to a wide range of theorems and constructions of rectilinear and circular figures, results of the geometric theory of proportions, and so on, as well as appreciation of the basic concerns in the deductive ordering of proofs. But regardless of whether Hippocrates did indeed work out the impressively formalized synthetic treatments given by Eudemus, rather than, say, a more loosely framed analytic treatment, his contribution to the study of these geometric problems is unquestionable.

PROBLEMS AND METHODS

This survey of the earliest efforts within the ancient tradition of problem solving has revealed the central importance of the work of Hippocrates of Chios. In particular, his studies of the cube duplication and of the quadrature of the circle and related figures laid the foundation for later researches. According to Proclus, he was also responsible for the first compilation of "Elements," that is, of a systematized presentation of propositions and proofs.[92] From the surviving fragments of his work, we may infer that this included substantial materials on the geometry of circles (cf. Euclid's Book III) as well as an exposition of the properties of plane figures dominated by the techniques of proportion theory (cf. Book VI), even if of course there were still lacking the more rigorous methods of limits and proportions (comparable to Books V and XII) due to Eudoxus.

Although stories like the Delian oracle on the doubling of the cube are often cited as suggesting an external motivation for the early studies of such problems, our examination of Eratosthenes' account of them and our inquiry into Hippocrates' work serve to discount this view. On the contrary, the interest in these problems can be seen as a fully natural outgrowth of researches within the geometric field. For instance, having cast the problem of squaring a given rectangle into the form of finding the mean proportional between its length and width, Hippocrates could be led to perceive that the finding of two mean proportionals is equivalent to increasing a given cube in a given ratio. In this way, his principal contribution to the study of the cube-duplication problem follows from the same techniques which characterize his study of plane figures. This study of quadratures leads readily to the further question as to whether a comparable construction for the circle might be produced. Here, of course, the quest for a solution fails, but not before the intriguing detour into the construction and quadrature of the lunules. To be sure, only three cases are actually found, although Hippocrates was surely aware of the defining conditions for an unlimited class of quadrable cases. The tantalizing result of having squared a certain lunule in combination with a circle would of course deceive no one into supposing that the circle itself had been squared. But in all these instances an important principle has been established: that curvilinear figures, specifically those associated with circular arcs, are not different in kind from rectilinear figures as far as their quadrability is concerned. Thus, when the later commentator Ammonius (5th century A.D.) presumes to have affirmed the impossibility of the circle quadrature through the intrinsic dissimilarity of the circular and the rectilinear, by analogy with the noncomparability of curvilinear and rectilinear angles, his disciple Simplicius brings forward the quadratures of the lunules as an immediate counterexample.[93]

In the matter of the lunules, the standard assumption is that Hippocrates consciously restricted the means allowable in their construction to the "Euclidean" devices of compass and straightedge. Although the cases he presents are indeed constructible by these means, the view flies in the face of his unexplicated introduction of a *neusis* in the construction of his third lunule. While of course the compass and straightedge were by this time long familiar as constructing instruments, and while the range of constructions effectible by these means was doubtless already appreciated to be quite extensive, nevertheless other means were then known, and still others would later be invented. The enterprise of discovering the solutions to problems could hardly be well served by the imposition of such a restriction at this early stage. Hippocrates must surely have been interested in finding constructions, using whatever means were available to him. The explicit restriction to one or another mode of construction is by its nature primarily a formal move, motivated by the urge to divide and classify the collected body of established results. Until the geometric corpus had attained a size and diversity meriting such efforts, there could hardly be much sense in engaging in these formal inquiries. While that level surely was reached around the time of Apollonius, it is open to debate whether it had already reached that level at the time of Euclid.[94] Applying this notion to the much earlier time of Hippocrates must be viewed as purely anachronistic.

There is a subtle distinction to be made here. The activity of seeking the

solutions to geometric problems is intrinsically formal, to the extent the constructions are to be provided within an implied context that imposes some restrictions on the available means and some measure of deductive ordering in the justifications. Thus, if one can rightly portray the most ancient traditions as empirical in that physical measures might be acceptable to justify certain claimed geometric results, then one cannot assign to these an involvement in problem solving. For in this environment any figure will be squarable, for instance, to within any desired perceptible degree of accuracy. By virtue of addressing the study of geometry in abstraction from overtly empirical measurements, the early Greek geometers would soon come upon configurations whose production proved intractable. For instance, the range of techniques known for the construction and study of similar rectilinear plane figures would be recognized as leading to no evident solution for the cube duplication or the circle quadrature. It is surely no mere coincidence, then, that Hippocrates was the first known systematizer of geometry as well as the first to formulate these constructions as problems, while such an awareness among earlier geometers, even Oenopides, is quite dubious.

On the other hand, the conscious restriction to a specific set of constructing techniques, like the compass and straightedge in sharp separation from others, would be premature at Hippocrates' time. This, I believe, may explain why the problems of rectifying and dividing circular arcs did not seem to attract interest until a century or more later. For a tradition of geometric practice which includes forms of protractors and cords among its tools might readily admit their abstract analogues in the more formalized study of figures, just as the use of sliding rulers gives rise to *neuses* and the use of compasses suggests the Euclidean postulate of circles.[95] Thus, the trisection of the angle became a problem only after a tightening of the restrictions on construction techniques, and hence only within a formally more sophisticated geometric field. By contrast, the problems of the cube and the circle present difficulties at the much earlier level of development. Indeed, one may anticipate that it was through the investigation of these and related problems that the clearer conceptions of the general nature of the problem-solving enterprise were to emerge.

NOTES TO CHAPTER 2

[1] Proclus, *In Euclidem*, ed. Friedlein, pp. 283, 333. On Oenopides and Hippocrates, cf. T. L. Heath, *History of Greek Mathematics*, I, pp. 174–202 and the articles in *Pauly Wissowa* and the *Dictionary of Scientific Biography*.

[2] Heath, *op. cit.*, p. 176; cf. Á. Szabó, *Anfänge der griechischen Mathematik*, III.19. I criticize this view in my "On the Early History of Axiomatics," p. 150.

[3] O. Neugebauer, *Mathematische Keilschrift-Texte*, I (*Quellen und Studien*, 1935, 3 : A), pp. 137–142. On the "rope stretchers" see Heath, *op. cit.*, pp. 121ff, and p. 178, where the term appears in connection with the mathematical talents of Democritus.

[4] This view is advocated with particular zeal by Á. Szabó, as in the work cited in note 2 above.

[5] Oenopides is reputed, for instance, to have discovered the obliquity of the ecliptic;

see Heath, *op. cit.*, p. 174. This is of course merely a testimony of the infancy of astronomical science among the Greeks in the 5th and 4th centuries, for such facts were certainly long familiar among the Mesopotamians. Of course, the specifically geometric approach to astronomy is characteristic of the Greek style, as contrasted with the arithmetic approach of the Mesopotamians. In this sense, the obliquity might indeed have been a new feature with Oenopides.

[6] Some of the instruments used in the practical work of geometry and architecture are named by Aristophanes (e.g., *Birds*, 1001 ff) and Plato (e.g., *Philebus* 51c, 56b, 62b). Among later developments one may note the elaborate sighting device described by Hero (in his *Dioptra, Opera* III) and the compass for drawing the conic sections devised by Isidore of Miletus (see the interpolation into Eutocius' commentary on Archimedes' *Sphere and Cylinder*, in Archimedes, *Opera*, ed. Heiberg, 2nd ed., III, p. 84). An instrument of the latter kind is described by al-Qūhī and other Arabic geometers; see the account by F. Woepcke, *L'Algèbre d'Omar*, 1851, p. 56n. In the same class may be mentioned the various mechanisms for the cube duplication used by "Plato," Eratosthenes, and Nicomedes (cf. Eutocius, *op. cit.*, pp. 56 ff, 90 ff, 98 ff; these are discussed in Chapters 3 and 6). A survey of the devices familiar in the ancient practical geometry is given by O. A. W. Dilke, *The Roman Land Surveyors*, Ch. 5. On the ancient architectural instruments of stone dressing, in particular, the set square, compass, ruler, and plumb line, see A. Orlandos, *Les Matériaux de construction et la technique architecturale des anciens grecs*, École française d'Athènes: *Travaux et Mémoires*, 16, pt. 2, 1968, pp. 59–69.

[7] See, for instance, Heath, *op. cit.*, I, pp. 218 ff; Becker, *Mathematisches Denken der Antike*, pp. 74 ff.

[8] Theon, *Expositio rerum utilium ad legendum Platonem*, ed. Hiller, p. 2. Plutarch, *Moralia* 386e, 579b. These passages have already been cited in Chapter 1.

[9] Cf. Archimedes, *Opera*, III, pp. 88–96.

[10] See, for instance, Heiberg's note in *Archimedes*, III, p. 89n; Heath, *History*, I, pp. 244 f; I. Thomas, *Greek Mathematical Works*, I, p. 256n; B. L. van der Waerden, *Science Awakening*, pp. 160 f. They all merely refer to von Wilamowitz as having established the inauthenticity of the document; see note 12 below.

[11] Eutocius in *Archimedes*, III, pp. 88–90 (i), 90 (ii), 90–92 (iii), 92–94 (iv), 94–96 (v).

[12] "Ein Weihgeschenk des Eratosthenes," *Göttinger Nachrichten*, 1894; repr. in *Kleine Schriften* II, 1971, pp. 48–70.

[13] *Collection*, ed. Hultsch (III) I, pp. 56–58.

[14] Eutocius, *op. cit.*, pp. 90, 96.

[15] See Fig. 1 (a-b) in Chapter 6 for Eratosthenes' construction.

[16] Eutocius, *op. cit.*, pp. 90, 96.

[17] For a gauge of the longevity of such a structure, one may consider the grave marker of Archimedes, all but lost within a century and a half of its foundation; the circumstances of Cicero's rediscovery of the monument in the 1st century B.C. are related in his *Tusculan Disputations*, V, 64ff.

[18] Recall that Hypatia, the learned daughter of Theon of Alexandria, died in 415 A.D. at the hands of a mob of rioting monks; cf. Heath, *History*, II, pp. 528 f, and P. Brown, *The World of Late Antiquity*, London, 1971, pp. 103 f.

[18a] See A. Fitzgerald, *The Letters of Synesius*, pp. 258–266 (esp. pp. 262–266); Fitzgerald provides a useful discussion of Synesius' career in the Introduction and notes.

[19] Cf., in particular, the prefaces to *Sphere and Cylinder* II and to *Conoids and Spheroids*; those to *Quadrature of the Parabola, Sphere and Cylinder* I and *Spiral Lines* provide a certain small amount of contextual information.

[20] Indeed, G. J. Toomer expresses misgivings over the general assumption that Alexandria was a monopolizing center of technical studies in this period; cf. his *Diocles*, p. 2. One notes that the prefaces to works by Diocles and by Apollonius are somewhat more ample in background information than those to the Archimedean writings.

[21] *Opera*, II, pp. 216–258. For accounts, see the Archimedes monographs by Heath, by Dijksterhuis, and by Schneider.

[22] In the 43 pages of the standard edition of the *Sand-Reckoner*, one finds about a dozen second-person forms (e.g., pp. 216, 218, 220, 234, 244, 246, 258); in Eutocius' text of Eratosthenes, a five-page document, there are three (pp. 90, 94, 96), not counting the epigram.

[23] For an account of the general background, see M. I. Finley, *Ancient Sicily*, New York, 1968, Ch. 9. I apply these materials toward the dating of the *Sand-Reckoner* in my "Archimedes and the *Elements*," pp. 234–238.

[24] Eutocius, *op. cit.*, pp. 88–90; cf. Heath, *History*, I, p. 245. I have inserted the reference letters.

[25] On Archytas, see Heath, *op. cit.*, pp. 213–216.

[26] Eutocius, *op. cit.*, p. 84.

[27] *Ibid.*, p. 96.

[28] *Expositio*, ed. Hiller, p. 2.

[29] *Moralia*, 386e, 579b–d. Cf. note 8 above.

[30] *Ibid.*, 718e f; *Vita Marcelli* xiv, 5. The latter contains an extended account of the life and exploits of Archimedes; see the Archimedes surveys cited in note 21 above.

[31] Details appear in Chapter 3.

[32] Heath, *History*, I, p. 246; van der Waerden, *op. cit.*, pp. 161–163. The latter seems to treat the story as a dramatization by Eratosthenes; but in a more recent effort, he wishes to treat it as a serious datum reflecting a ritual tradition within early mathematics; see his "On Pre-Babylonian Mathematics (II)," *Archive for History of Exact Sciences*, 1980, 23, pp. 37 f. This thesis is advocated by A. Seidenberg in several articles, particularly, "The Ritual Origin of Geometry," *Archive for History of Exact Sciences*, 1963, 1, pp. 488–527.

[33] Eutocius, *op. cit.*, p. 88; to be discussed further below.

[34] Note in particular Plato's criticisms of geometric research and his recommendations for mathematical education in *Republic*, Books VI–VII.

[35] Cf. *Post. Ana.* I, 7 and *Meta.* M.3. Passages on the ordering and nature of the sciences are collected and discussed by Heath, *Mathematics in Aristotle*, pp. 4–12, 46, 59 f, 64–67, 225.

[36] Eutocius, *op. cit.*, p. 88.

[37] *Ibid.*, pp. 88–90.

[38] A form of the proof, applied to the more general case of similar solids, appears in Euclid's *Elements* XI, 33 (porism).

[39] *In Euclidem*, p. 213. On the method of analysis, see Chapter 3.

[40] *De Anima* II, 2, 413 a 16–20. This passage is noted, if briefly, in most editions of the work; cf. W. D. Ross, Aristotle's *De Anima*, Oxford, 1961, p. 217, and also Heath, *Mathematics in Aristotle*, pp. 191–193. Most commentators, however, attempt to construe it as a specific reference to the Euclidean constructions of either or both of the two problems, rather than as a statement of the reduction of the one to the other.

[41] This fragment, preserved in Simplicius' commentary on Aristotle's *Physics*, is discussed further in this and the next chapter.

[42] On the ancient Egyptian and Mesopotamian mathematical traditions, see O. Neugebauer, *Exact Sciences in Antiquity*, Ch. 2, 4; and van der Waerden, *op. cit.*, Ch. 1, 2. On the thesis of the ritual origins of technique, see note 32 above. Establishing a perspective on the relation of geometry and philosophy in the pre-Euclidean period is an aim of Chapter 3 below.

[43] *In Euclidem*, pp. 422 f. Proclus cites as an example of a theorem on the circle quadrature Archimedes' *Dimension of the Circle*, Prop. 1; see Chapter 5.

[44] Problem 48; cf. the edition by A. Chace *et al.*, 1927–29, II, Plate 70. For a discussion, see R. Gillings, *Mathematics in the Time of the Pharaohs*, 1972, pp. 139–146, and H. Engels, "Quadrature of the Circle in Ancient Egypt," *Historia Mathematica*, 1977, 4, pp. 137–140. The rule is equivalent to approximating π as 256/81, that is, 3 13/81 or a bit less than 3 1/6

[45] That this relation was recognized is indicated by the use of the correct rule for the volume of the truncated pyramid in the Moscow Papyrus (c. 1600 B.C.); see van der Waerden, *op. cit.*, pp. 34 f. For this would seem to require knowledge of the 1 : 3 ratio between the pyramid and the corresponding parallelepiped. Physical procedures for the measurement of irregular surfaces and solids are found in the Greek metrical tradition; cf. Hero, *Metrica* I, 39 and II, 20.

[46] *The Birds*, lin. 1001–1005; cf. Heath, *History*, I, pp. 220 f and I. Thomas, *op. cit.*, pp. 308 f. Thomas translates *kampylos kanōn* (literally, "curved ruler") as "flexible rod," and this may be supported in the light of Aristotle's mention of the "leaden rule" used by architects (*Nic. Eth.* 1137 b 30) and by Diocles' use of a flexible ruler of horn for the purposes of curve plotting; see Toomer, *Diocles*, pp. 159 f. H. Mendell informs me of an alternative rendering of the passage which conforms better both to sense and to the standard punctuation: "If I lay out the ruler and insert this curved compass from above—do you see?...I shall measure with the straight ruler by laying (it) out..." In this way, "curved" modifies "compass" (cf. Aristophanes, *Clouds*, 178) rather than "ruler". This of course does not affect the later testimonia to the use of flexible rulers in geometric practice.

[47] Cf. Heath, *loc. cit.* The passage itself gives no fully clear notion of just *what* Meton is up to, and perhaps this is part of the joke.

[48] *Physics* 185 a 16; *Sophistical Refutations* 171 b 15; cf. Heath, *Mathematics in Aristotle*, pp. 33–36.

[49] Simplicius, *In Physica*, ed. Diels, I, p. 60. The fragment from Eudemus occupies pp. 61–68, followed by an assessment on pp. 68 f; Simplicius takes up other versions of the circle quadrature on pp. 53–60.

[50] *Collection* (V), I, pp. 340–342. See my discussion in "Infinity and Continuity," pp. 127 ff.

[51] On this terminology, see my "Archimedes and the *Elements*," pp. 240n, 254n, 264; and my "Archimedes and the Pre-Euclidean Proportion Theory," p. 196.

[52] On the Eudoxean methods and their attribution via Archimedes' *Quadrature of the Parabola* and *Sphere and Cylinder* I, see my "Archimedes and the Pre-Euclidean Proportion Theory," pp. 194 ff.

[53] For a general account of Antiphon's thought, see W. K. C. Guthrie, *A History of Greek Philosophy*, III, pp. 285–294. Simplicius' passage on his circle quadrature (*In Physica*, pp. 54 f) would appear to derive from Eudemus, for his specific critique of this argument is cited (*ibid.*, p. 55). The same critique, if without express citation of Eudemus, is given also by Themistius (4th century A.D., *In Physica*, ed. H. Schenkl, *CAG* 5, Pt. II, 1900, pp. 3 f), namely, that it violates the principle of the unlimited divisibility of continuous magnitude. But Simplicius also cites an alternative critique by Alexander, so that he may well have derived his account of Antiphon via Alexander, rather than directly from Eudemus.

[54] Simplicius, *op. cit.*, pp. 54 f.

[55] *Ibid.*, p. 55 (see note 53 above). On Protagoras' treatment of this issue, see Aristotle's *Metaphysics* B, 2, 997 b 32 and its discussion by Guthrie, *op. cit.*, II, p. 486 and III, p. 267. On Democritus' work, see Heath, *History*, I, pp. 176–181 and Guthrie, *op. cit.*, II, Ch. viii, especially pp. 487 f.

[56] Cf. Eudemus' view, cited in note 53 above.

[57] See, for instance, Heath, *History*, I, p. 224.

[58] See my "Infinity and Continuity," pp. 133 f. The summation theorem for the finite case is given in *Elements* V, 12.

[59] Hippocrates seems to appeal to a concept of proper "parts" as the basis of his notion of ratio; cf. Simplicius, *op. cit.*, p. 61 and the discussion by Heath, *History*, I, pp. 187–191. This could provide the precedent for a theory of proportions for commensurable magnitudes; but the extension to incommensurable magnitudes, as in the Eudoxean theory, had not yet been worked out at Hippocrates' time. See my "Archimedes and the Pre-Euclidean Proportion Theory," and Chapter 3.

[60] Further examples are found in Pappus' measurements of figures bounded by spirals; see Chapter 5.

[61] Plutarch, *Moralia* 607 f (cf. Diels and Kranz, A 38, II, p. 14). The view that this might signify a *writing* by Anaxagoras on circle quadrature is discounted by J. Burnet (*Early Greek Philosophy*, 4th ed., p. 257) and by Guthrie (*op. cit.*, II, p. 270).

[62] *In Euclidem*, pp. 65 f. Vitruvius also mentions Anaxagoras for a study of perspective; cf. Heath, *History*, I, pp. 172-174. For a general account of Anaxagoras, see Guthrie, *op. cit.*, II, Ch. iv.

[63] *Prior Analytics* II, 25; cf. the passages cited in note 48 above.

[64] See note 49 above.

[65] Simplicius, *op. cit.*, pp. 56 f.

[66] This readily suggests an alternative construction in which the right triangle may be scalene rather than isosceles. Here too the two lunules will sum to the area of the triangle.

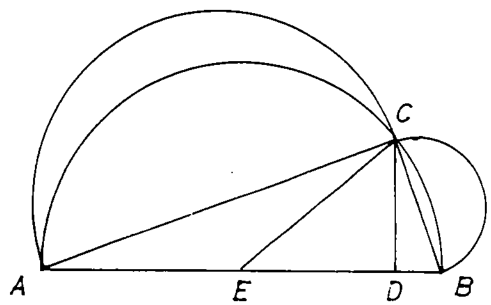


Figure 8

Simplicius' text does not mention this variant, but it has slipped into modern accounts as if by Hippocrates himself. M. Simon notes having failed to discover this variant in works prior to the late 17th century ("Lunulae Hippocratis," *Archiv für Mathematik und Physik*, 1904–05, 8, p. 269). But a medieval precedent may be found in treatments by ibn al-Haytham and Joannes de Muris; cf. M. Clagett, *Archimedes in the Middle Ages*, III, pp. 1315n, 1321f. One notes that since the small segments over the sides AC, CB are not similar, neither are the lunules; hence the lunules will not be in the ratio of the similar triangles ACD, DCB, respectively, but rather, lunule AC will be less and lunule CB greater than its corresponding triangle. Ibn al-Haytham observes that the lunules will be equal to the respective triangles ACE, ECB (where E is the center of the circle) when the arcs AC and CB are equal. The relation is of course not general, for in the scalene case the lunule over the greater leg will be greater than that over the smaller leg; the maximal discrepancy occurs for central angle CEB having sine equal to $2/\pi$ (approx. 39½°), where the lunules will be in the ratio 3.1616... : 1. [This is the configuration shown in Fig. 8.]

67 For a review of modern interpretations, particularly those of Tannery and Björnbo, see Heath, *History*, I, p. 196n and Thomas, *op. cit.*, I, p. 310n. The opinion of Alexander is cited by Simplicius, *op. cit.*, pp. 57 f, 60, while his own view appears on pp. 60, 69.

68 For full accounts, see Heath, *History*, I, pp. 191–200 and Thomas, *op. cit.*, I, pp. 234–253. A major interpretive issue here centers on the segregation of Eudemus' text from Simplicius' interpolations. Proposals have been offered by Allman, Diels (1882), Tannery (1883), Rudio (1907), and Becker (1936). For references, see Thomas, *op. cit.*, p. 237n. Rudio's translation and commentary contain many useful insights ("Bericht des Simplicius...," *Bibliotheca Mathematica*, 3_3, 1903, pp. 7–62); see also my "Infinity and Continuity," pp. 127–130. Thomas follows Rudio's abridgment, so that the reader interested in judging for himself must go back to Diels' text.

69 This is done by applying the principle that in right triangles the square on the longest side equals the sum of those on the other two sides, but in obtuse triangles it is greater and in acute ones less; cf. *Elements* I, 47 and II, 12–13. In the present case, since angle BAG is obtuse, $BG^2 > BA^2 + AG^2 = 2\,GD^2$, whence $BD^2\ (= 3\,GD^2) < BG^2 + GD^2$, so that angle BGD is acute. Hence, arc BGD is greater than a semicircle, since the angles inscribed in segments are proportional to the corresponding supplementary arcs (cf. Simplicius, *op. cit.*, p. 60).

70 Specifically, this text includes archaic usage of the form "the line *on which* AB," where later writers would invariably say "the line AB"; see below, note 90.

71 Signs of such minor textual corruption appear elsewhere, for instance, in the treatment of case (iii) where certain steps in the construction are out of the proper order (Simplicius, *op. cit.*, p. 65); cf. Thomas, *op. cit.*, p. 244n.

72 On the method of "analysis and synthesis," standard in the later problem-solving tradition, see Chapter 3.

73 For if one introduces line HZK, the angles at K and B will be equal; hence, Z is the vertex of the isosceles triangle ZKB.

74 Simplicius, *op. cit.*, p. 64.

75 See, in particular, Chapters 5, 6, and 7. A survey of ancient *neuses* is given by R. Böker in *Pauly Wissowa*, Suppl. ix, 1962, col. 415–461. Cf. note 86 below.

76 Simplicius, *op. cit.*, p. 66. The text adds, superfluously, "and the rectilinear figure is [the meniscus] with the two segments, but without the three."

77 *Ibid.*, p. 67. Writing "[he squared]" after "seeing that" (or "since in fact," in my rendering of *eiper kai*), Heath assumes an ellipsis, and he is well supported in view of the grammatical role of *eiper* as a subordinating conjunction. This entails, however, the fallacy of concluding the general from the particular. Wishing to avoid this, Tannery and Björnbo have proposed excising "every" and "seeing that" as interpolations (see Heath, *loc. cit.*).

78 Simplicius, *op. cit.*, p. 68.

79 *Ibid.*, p. 69; cf. Aristotle, *Physics* 185 a 16 (note 48 above).

80 Note that lines 0C and 0′C′ will be tangent to the circular arc.

81 This is under the restriction that θ be less than $180/m$, for m the number of elements in the polygonal arc.

82 Cf. Heath, *History*, I, p. 200.

83 E. Landau, "Ueber quadrierbare Kreisbogenzweiecke," *Sitzungsberichte*, Berlin Math. Ges., 1902–03, pp. 1–6 (suppl. to *Archiv für Math. und Phys.*, 4, 1902–03); cited by F. Enriques (ed.), *Fragen der Elementargeometrie*, 1923, II, pp. 304–308. Tschakaloff, "Beitrag zum Problem der quadrierbaren Kreisbogenzweiecke," *Math. Zeitschrift*, 1929, 30, pp. 552–559; this and other efforts are cited by L. Bieberbach (cf. note 89 below), pp. 140 f, 159.

84 References are given by A. D. Steele, "Zirkel und Lineal," *Quellen und Studien*, 1936, 3 : B, p. 317. The algebraic condition to which the case (1, 9) reduces has some constructible roots, but these are imaginary; its real roots are nonconstructible.

85 See Heath, *Euclid*, I, pp. 386 f and Chapter 3 below.

86 Steele provides an alternative planar construction of this *neusis*, following the model of a result from Apollonius' *Neuses* as discussed by Pappus (*op. cit.*, pp. 319–322). But Steele goes on to insist, on textual grounds, that the Hippocrates–Eudemus passage, as extant, could not have intended any such alternative method in substitution for the *neusis*.

87 See Heath, *Archimedes*, pp. ci–ciii, who follows Zeuthen in supposing that Archimedes must have understood alternative constructions of the *neuses*. But Dijksterhuis (*Archimedes*, pp. 138 f) insists, I believe correctly, that Archimedes accepted the *neuses* in their own right; see Chapter 5.

[88] As θ continually decreases from $180/(m + n)$ to 0, the ratio X : Y continually increases from 1 : 1 to $m : n$. Since the ratio $\sqrt{m} : \sqrt{n}$ lies within this range, X : Y will assume this value for some angle within the corresponding domain of θ. One may observe that Hippocrates' *neusis* for the case (2, 3), when X = Y, yields a construction for the regular pentagon. Some have wished to assign a method of this sort to the ancient Pythagoreans; but, unfortunately, there is no documentary support for this claim (cf. R. Böker, "Winkelteilung," pp. 137 f).

[89] In this connection, Steele notes Vieta's treatment of the case (1, 4); see *op. cit.*, p. 318. *Neuses* may be used for the solution of third-order problems like the cube duplication and the angle trisection; see Chapters 5, 6, and 7. For an account of this construction technique, see L. Bieberbach, *Theorie der geometrischen Konstruktionen*, Basel, 1952, Ch. 16, 17.

[90] See note 70 above. The language of proportions here is also archaic. Eudemus' text speaks of the proportion A : B = G : D by the expression "A is greater than B by the same part that G is greater than D" (Simplicius, *op. cit.*, p. 60). This may be compared with the phrasing in Archytas (fr. 2; Diels–Kranz, 47 B 2, I, pp. 435 f) and in the pseudo-Aristotelian *Mechanics*, Ch. 20. By contrast, Euclid, Archimedes, and other late writers will say "A has the same ratio to B that G has to D," or more simply, "A is to B as G is to D."

[91] Note, for instance, that the commentators commonly speak of "those in the circle of..." (*hoi peri*), thus leaving it unclear whether they refer to the master of the tradition or to his followers. Eudemus himself was evidently willing sometimes to make historical claims on the basis of inference. He says of Thales, for instance, that he knew a certain theorem on congruent triangles (*Elem.* I, 26), for he "must have used this" for determining the distance of ships at sea, as he is reported to have done (cf. Proclus, *In Euclidem*, p. 352). The difficulty of discerning the actual basis of this and other claims by Eudemus has thus led to a spectrum of widely divergent views on the achievements of Thales. For references, see W. Burkert, *Lore and Science in Ancient Pythagoreanism*, 1972, pp. 415–417.

[92] Proclus, *In Euclidem*, p. 66.

[93] Simplicius, *op. cit.*, pp. 59 f. Ammonius here refers to the "horn angle" defined by the space between the circle and its tangent (cf. *Elem.* III, 16 and Heath, *Euclid*, II, pp. 39 ff). Proclus makes several references to this and related mixed and curvilinear angles (cf. *In Eucl.*, pp. 122, 127, 134). These figure within Philoponus' discussion of Bryson's circle quadrature (see Chapter 3).

[94] This will be discussed in more detail in Chapters 7 and 8.

[95] In particular, the availability of "curved (flexible) rulers" suggests an extremely simple practical procedure for these constructions; see note 46 above and Chapter 3.

CHAPTER 3

The Geometers in Plato's Academy

Through Hippocrates' efforts a start was made toward the investigation of geometric problems. But the first discovery of actual solutions for more difficult problems, like the circle quadrature and the cube duplication, was made by geometers in the generations after him. Hippocrates' reduction of the latter problem, for instance, to the form of finding two mean proportionals between two given lines served as the basis for the successful constructions by Archytas, Eudoxus, and Menaechmus. At the same time, new geometric methods were introduced, in particular the use of special curves generated through the sectioning of solids or through the geometric conception of mechanical motions. Furthermore, Hippocrates' precedent for the "reduction" of one problem to another could lead others to the articulation of the versatile technique of geometric "analysis."

During the 4th century B.C., Plato's Academy in Athens became a center for geometric studies. This is reflected, as we have seen, in such legendary accounts as that of the Delian oracle, portraying Plato's interaction with mathematicians. Although later authorities sometimes assign to him the discovery of specific technical results, like a method for the duplication of the cube, the veracity of such reports is dubious and generally discounted. In the reasonable assessment by Proclus, it was rather through the special position he afforded mathematical study within his program of philosophical education, elaborating a plan initiated by the older Pythagoreans, that Plato encouraged mathematical research, as well as through his incorporation of technical examples into his writings to illustrate points of method.[1] Plato appears to have been much impressed by the technical rigor of some of the older geometers, notably Theodorus of Cyrene, a contem-