(i) Mechanics : Centres of Gravity

(i.) *Postulates*

Archim. *De Plan. Aequil.*, Deff., Archim. ed. Heiberg ii. 124. 3–126. 3

α΄. Αἰτούμεθα τὰ ἴσα βάρεα ἀπὸ ἴσων μακέων ἰσορροπεῖν, τὰ δὲ ἴσα βάρεα ἀπὸ τῶν ἀνίσων μακέων μὴ ἰσορροπεῖν, ἀλλὰ ῥέπειν ἐπὶ τὸ βάρος τὸ ἀπὸ τοῦ μείζονος μάκεος.

(*b*)

$$x=(\tfrac{1}{3}+\tfrac{1}{4})(Y+y) \quad . \quad . \quad . \quad . \quad (4)$$
$$y=(\tfrac{1}{4}+\tfrac{1}{5})(W+w) \quad . \quad . \quad . \quad . \quad (5)$$
$$w=(\tfrac{1}{5}+\tfrac{1}{6})(Z+z) \quad . \quad . \quad . \quad . \quad (6)$$
$$z=(\tfrac{1}{6}+\tfrac{1}{7})(X+x) \quad . \quad . \quad . \quad . \quad (7)$$

The second part of the epigram states that

$$X+Y=\text{a rectangular number} \quad . \quad . \quad (8)$$
$$Z+W=\text{a triangular number} \quad . \quad . \quad . \quad (9)$$

This was solved by J. F. Wurm, and the solution is given by A. Amthor, *Zeitschrift für Math. u. Physik.* (*Hist.-litt. Abtheilung*), xxv. (1880), pp. 153-171, and by Heath, *The Works of Archimedes*, pp. 319-326. For reasons of space, only the results can be noted here.

Equations (1) to (7) give the following as the values of the unknowns in terms of an unknown integer n :

$X=10366482n$	$x=7206360n$
$Y=7460514n$	$y=4893246n$
$Z=4149387n$	$z=5439213n$
$W=7358060n$	$w=3515820n.$

We have now to find a value of n such that equation (9) is also satisfied—equation (8) will then be simultaneously satisfied. Equation (9) means that

$$Z+W=\frac{p(p+1)}{2},$$

where p is some positive integer, or

$$(4149387+7358060)n=\frac{p(p+1)}{2},$$

(i) Mechanics : Centres of Gravity

(i.) *Postulates*

Archimedes, *On Plane Equilibriums*,[a] Definitions, Archim. ed. Heiberg ii. 124. 3–126. 3

1. I postulate that equal weights at equal distances balance, and equal weights at unequal distances do not balance, but incline towards the weight which is at the greater distance.

i.e.

$$2471\,.\,4657n=\frac{p(p+1)}{2}.$$

This is found to be satisfied by

$$n=3^3\,.\,4349,$$

and the final solution is

$X=1217263415886$	$x=846192410280$
$Y=876035935422$	$y=574579625058$
$Z=487233469701$	$z=638688708099$
$W=864005479380$	$w=412838131860$

and the total is 5916837175686.

If equation (8) is taken to be that $X+Y=$ a square number, the solution is much more arduous ; Amthor found that in this case,

$$W=1598\ \langle\overline{206541}\rangle,$$

where $\langle\overline{206541}\rangle$ means that there are 206541 more digits to follow, and the whole number of cattle $=7766\ \langle\overline{206541}\rangle$. Merely to write out the eight numbers, Amthor calculates, would require a volume of 660 pages, so we may reasonably doubt whether the problem was really framed in this more difficult form, or, if it were, whether Archimedes solved it.

[a] This is the earliest surviving treatise on mechanics ; it presumably had predecessors, but we may doubt whether mechanics had previously been developed by rigorous geometrical principles from a small number of assumptions. References to the principle of the lever and the parallelogram of velocities in the Aristotelian *Mechanics* have already been given (vol. i. pp. 430-433).

β΄. εἴ κα βαρέων ἰσορροπεόντων ἀπό τινων μακέων ποτὶ τὸ ἕτερον τῶν βαρέων ποτιτεθῇ, μὴ ἰσορροπεῖν, ἀλλὰ ῥέπειν ἐπὶ τὸ βάρος ἐκεῖνο, ᾧ ποτετέθη.

γ΄. Ὁμοίως δὲ καί, εἴ κα ἀπὸ τοῦ ἑτέρου τῶν βαρέων ἀφαιρεθῇ τι, μὴ ἰσορροπεῖν, ἀλλὰ ῥέπειν ἐπὶ τὸ βάρος, ἀφ᾽ οὗ οὐκ ἀφῃρέθη.

δ΄. Τῶν ἴσων καὶ ὁμοίων σχημάτων ἐπιπέδων ἐφαρμοζομένων ἐπ᾽ ἄλλαλα καὶ τὰ κέντρα τῶν βαρέων ἐφαρμόζει ἐπ᾽ ἄλλαλα.

ε΄. Τῶν δὲ ἀνίσων, ὁμοίων δέ, τὰ κέντρα τῶν βαρέων ὁμοίως ἐσσεῖται κείμενα. ὁμοίως δὲ λέγομες σαμεῖα κέεσθαι ποτὶ τὰ ὁμοῖα σχήματα, ἀφ᾽ ὧν ἐπὶ τὰς ἴσας γωνίας ἀγόμεναι εὐθεῖαι ποιέοντι γωνίας ἴσας ποτὶ τὰς ὁμολόγους πλευράς.

ϛ΄. Εἴ κα μεγέθεα ἀπό τινων μακέων ἰσορροπέωντι, καὶ τὰ ἴσα αὐτοῖς ἀπὸ τῶν αὐτῶν μακέων ἰσορροπήσει.

ζ΄. Παντὸς σχήματος, οὗ κα ἁ περίμετρος ἐπὶ τὰ αὐτὰ κοίλα ᾖ, τὸ κέντρον τοῦ βάρεος ἐντὸς εἶμεν δεῖ τοῦ σχήματος.

(ii.) *Principle of the Lever*

Ibid., Props. 6 et 7, Archim. ed. Heiberg ii. 132. 13–138. 8

ϛ΄

Τὰ σύμμετρα μεγέθεα ἰσορροπέοντι ἀπὸ μακέων ἀντιπεπονθότως τὸν αὐτὸν λόγον ἐχόντων τοῖς βάρεσιν.

Ἔστω σύμμετρα μεγέθεα τὰ Α, Β, ὧν κέντρα τὰ Α, Β, καὶ μᾶκος ἔστω τι τὸ ΕΔ, καὶ ἔστω, ὡς τὸ Α ποτὶ τὸ Β, οὕτως τὸ ΔΓ μᾶκος ποτὶ τὸ ΓΕ

2. If weights at certain distances balance, and something is added to one of the weights, they will not remain in equilibrium, but will incline towards that weight to which the addition was made.

3. Similarly, if anything be taken away from one of the weights, they will not remain in equilibrium, but will incline towards the weight from which nothing was subtracted.

4. When equal and similar plane figures are applied one to the other, their centres of gravity also coincide.

5. In unequal but similar figures, the centres of gravity will be similarly situated. By points similarly situated in relation to similar figures, I mean points such that, if straight lines be drawn from them to the equal angles, they make equal angles with the corresponding sides.

6. If magnitudes at certain distances balance, magnitudes equal to them will also balance at the same distances.

7. In any figure whose perimeter is concave in the same direction, the centre of gravity must be within the figure.

(ii.) *Principle of the Lever*

Ibid., Props. 6 and 7, Archim. ed. Heiberg ii. 132. 13–138. 8

Prop. 6

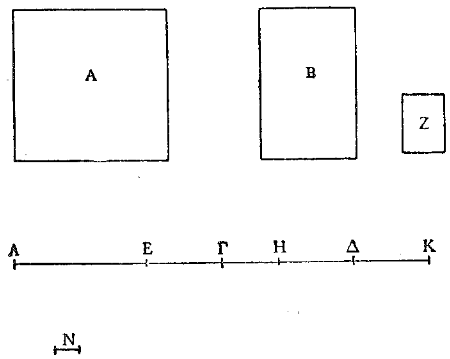
Commensurable magnitudes balance at distances reciprocally proportional to their weights.

Let A, B be commensurable magnitudes with centres [of gravity] A, B, and let EΔ be any distance, and let

$$A : B = \Delta\Gamma : \Gamma E;$$

μᾶκος· δεικτέον, ὅτι τοῦ ἐξ ἀμφοτέρων τῶν Α, Β συγκειμένου μεγέθεος κέντρον ἐστὶ τοῦ βάρεος τὸ Γ.

Ἐπεὶ γάρ ἐστιν, ὡς τὸ Α ποτὶ τὸ Β, οὕτως τὸ ΔΓ ποτὶ τὸ ΓΕ, τὸ δὲ Α τῷ Β σύμμετρον, καὶ τὸ ΓΔ ἄρα τῷ ΓΕ σύμμετρον, τουτέστιν εὐθεῖα τᾷ εὐθείᾳ· ὥστε τῶν ΕΓ, ΓΔ ἐστι κοινὸν μέτρον. ἔστω δὴ τὸ Ν, καὶ κείσθω τᾷ μὲν ΕΓ ἴσα ἑκατέρα τᾶν ΔΗ, ΔΚ, τᾷ δὲ ΔΓ ἴσα ἁ ΕΛ. καὶ ἐπεὶ ἴσα



ἁ ΔΗ τᾷ ΓΕ, ἴσα καὶ ἁ ΔΓ τᾷ ΕΗ· ὥστε καὶ ἁ ΛΕ ἴσα τᾷ ΕΗ. διπλασία ἄρα ἁ μὲν ΛΗ τᾶς ΔΓ, ἁ δὲ ΗΚ τᾶς ΓΕ· ὥστε τὸ Ν καὶ ἑκατέραν τᾶν ΛΗ, ΗΚ μετρεῖ, ἐπειδήπερ καὶ τὰ ἡμίσεα αὐτᾶν. καὶ ἐπεί ἐστιν, ὡς τὸ Α ποτὶ τὸ Β, οὕτως ἁ ΔΓ ποτὶ ΓΕ, ὡς δὲ ἁ ΔΓ ποτὶ ΓΕ, οὕτως ἁ ΛΗ ποτὶ ΗΚ—διπλασία γὰρ ἑκατέρα ἑκατέρας—καὶ ὡς ἄρα τὸ Α ποτὶ τὸ Β, οὕτως ἁ ΛΗ ποτὶ ΗΚ. ὁσαπλασίων δέ ἐστιν ἁ ΛΗ τᾶς Ν, τοσαυ-

it is required to prove that the centre of gravity of the magnitude composed of both A, B is Γ.

Since $$A : B = \Delta\Gamma : \Gamma E,$$

and A is commensurate with B, therefore ΓΔ is commensurate with ΓΕ, that is, a straight line with a straight line [Eucl. x. 11]; so that ΕΓ, ΓΔ have a common measure. Let it be N, and let ΔΗ, ΔΚ be each equal to ΕΓ, and let ΕΛ be equal to ΔΓ. Then since $\Delta H = \Gamma E$, it follows that $\Delta\Gamma = EH$; so that $\Lambda EE = H$. Therefore $\Lambda H = 2\Delta\Gamma$ and $HK = 2\Gamma E$; so that N measures both ΛΗ and ΗΚ, since it measures their halves [Eucl. x. 12]. And since

$$A : B = \Delta\Gamma : \Gamma E,$$

while $$\Delta\Gamma : \Gamma E = \Lambda H : HK—$$

for each is double of the other—

therefore $$A : B = \Lambda H : HK.$$

Now let Z be the same part of A as N is of ΛΗ;

ταπλασίων ἔστω καὶ τὸ Α τοῦ Ζ· ἔστιν ἄρα, ὡς ἁ ΛΗ ποτὶ Ν, οὕτως τὸ Α ποτὶ Ζ. ἔστι δὲ καί, ὡς ἁ ΚΗ ποτὶ ΛΗ, οὕτως τὸ Β ποτὶ Α· δι' ἴσου ἄρα ἐστίν, ὡς ἁ ΚΗ ποτὶ Ν, οὕτως τὸ Β ποτὶ Ζ· ἰσάκις ἄρα πολλαπλασίων ἐστὶν ἁ ΚΗ τᾶς Ν καὶ τὸ Β τοῦ Ζ. ἐδείχθη δὲ τοῦ Ζ καὶ τὸ Α πολλαπλάσιον ἐόν· ὥστε τὸ Ζ τῶν Α, Β κοινόν ἐστι μέτρον. διαιρεθείσας οὖν τᾶς μὲν ΛΗ εἰς τὰς τᾷ Ν ἴσας, τοῦ δὲ Α εἰς τὰ τῷ Ζ ἴσα, τὰ ἐν τᾷ ΛΗ τμάματα ἰσομεγέθεα τᾷ Ν ἴσα ἐσσεῖται τῷ πλήθει τοῖς ἐν τῷ Α τμαμάτεσσιν ἴσοις ἐοῦσιν τῷ Ζ. ὥστε, ἂν ἐφ' ἕκαστον τῶν τμαμάτων τῶν ἐν τᾷ ΛΗ ἐπιτεθῇ μέγεθος ἴσον τῷ Ζ τὸ κέντρον τοῦ βάρεος ἔχον ἐπὶ μέσου τοῦ τμάματος, τά τε πάντα μεγέθεα ἴσα ἐντὶ τῷ Α, καὶ τοῦ ἐκ πάντων συγκειμένου κέντρον ἐσσεῖται τοῦ βάρεος τὸ Ε· ἄρτιά τε γάρ ἐστι τὰ πάντα τῷ πλήθει, καὶ τὰ ἐφ' ἑκάτερα τοῦ Ε ἴσα τῷ πλήθει διὰ τὸ ἴσαν εἶμεν τὰν ΛΕ τᾷ ΗΕ.

Ὁμοίως δὲ δειχθήσεται, ὅτι κἄν, εἴ κα ἐφ' ἕκαστον τῶν ἐν τᾷ ΚΗ τμαμάτων ἐπιτεθῇ μέγεθος ἴσον τῷ Ζ κέντρον τοῦ βάρεος ἔχον ἐπὶ τοῦ μέσου τοῦ τμάματος, τά τε πάντα μεγέθεα ἴσα ἐσσεῖται τῷ Β, καὶ τοῦ ἐκ πάντων συγκειμένου κέντρον τοῦ βάρεος ἐσσεῖται τὸ Δ· ἐσσεῖται οὖν τὸ μὲν Α ἐπικείμενον κατὰ τὸ Ε, τὸ δὲ Β κατὰ τὸ Δ. ἐσσεῖται δὴ μεγέθεα ἴσα ἀλλάλοις ἐπ' εὐθείας κείμενα, ὧν τὰ κέντρα τοῦ βάρεος ἴσα ἀπ' ἀλλάλων διέστακεν, [συγκείμενα][1] ἄρτια τῷ πλήθει· δῆλον οὖν, ὅτι τοῦ ἐκ πάντων συγκειμένου μεγέθεος κέντρον ἐστὶ τοῦ βάρεος ἁ διχοτομία τᾶς εὐθείας τᾶς ἐχούσας τὰ κέντρα τῶν μέσων μεγεθέων. ἐπεὶ δ' ἴσαι ἐντὶ

then ΑΗ : Ν = Α : Ζ. [Eucl. v., Def. 5

And ΚΗ : ΛΗ = Β : Α ; [Eucl. v. 7, coroll.

therefore, *ex aequo*,

ΚΗ : Ν = Β : Ζ ; [Eucl. v. 22

therefore Ζ is the same part of Β as Ν is of ΚΗ. Now Α was proved to be a multiple of Ζ ; therefore Ζ is a common measure of Α, Β. Therefore, if ΛΗ is divided into segments equal to Ν and Α into segments equal to Ζ, the segments in ΛΗ equal in magnitude to Ν will be equal in number to the segments of Α equal to Ζ. It follows that, if there be placed on each of the segments in ΛΗ a magnitude equal to Ζ, having its centre of gravity at the middle of the segment, the sum of the magnitudes will be equal to Α, and the centre of gravity of the figure compounded of them all will be Ε ; for they are even in number, and the numbers on either side of Ε will be equal because ΛΕ = ΗΕ. [Prop. 5, coroll. 2.]

Similarly it may be proved that, if a magnitude equal to Ζ be placed on each of the segments [equal to Ν] in ΚΗ, having its centre of gravity at the middle of the segment, the sum of the magnitudes will be equal to Β, and the centre of gravity of the figure compounded of them all will be Δ [Prop. 5, coroll. 2]. Therefore Α may be regarded as placed at Ε, and Β at Δ. But they will be a set of magnitudes lying on a straight line, equal one to another, with their centres of gravity at equal intervals, and even in number ; it is therefore clear that the centre of gravity of the magnitude compounded of them all is the point of bisection of the line containing the centres [of gravity] of the middle magnitudes [from Prop. 5, coroll. 2].

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[1] συγκείμενα om. Heiberg.

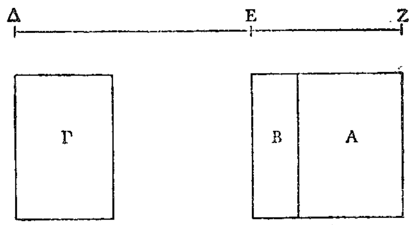
ἁ μὲν ΛΕ τᾷ ΓΔ, ἁ δὲ ΕΓ τᾷ ΔΚ, καὶ ὅλα ἄρα ἁ ΛΓ ἴσα τᾷ ΓΚ· ὥστε τοῦ ἐκ πάντων μεγέθεος κέντρον τοῦ βάρεος τὸ Γ σαμεῖον. τοῦ μὲν ἄρα Α κειμένου κατὰ τὸ Ε, τοῦ δὲ Β κατὰ τὸ Δ, ἰσορροπησοῦντι κατὰ τὸ Γ.

ζʹ

Καὶ τοίνυν, εἴ κα ἀσύμμετρα ἔωντι τὰ μεγέθεα, ὁμοίως ἰσορροπησοῦντι ἀπὸ μακέων ἀντιπεπονθότως τὸν αὐτὸν λόγον ἐχόντων τοῖς μεγέθεσιν.

Ἔστω ἀσύμμετρα μεγέθεα τὰ ΑΒ, Γ, μάκεα δὲ τὰ ΔΕ, ΕΖ, ἐχέτω δὲ τὸ ΑΒ ποτὶ τὸ Γ τὸν αὐτὸν λόγον, ὃν καὶ τὸ ΕΔ ποτὶ τὸ ΕΖ μᾶκος· λέγω, ὅτι τοῦ ἐξ ἀμφοτέρων τῶν ΑΒ, Γ κέντρον τοῦ βάρεός ἐστι τὸ Ε.

Εἰ γὰρ μὴ ἰσορροπήσει τὸ ΑΒ τεθὲν ἐπὶ τῷ Ζ τῷ Γ τεθέντι ἐπὶ τῷ Δ, ἤτοι μεῖζόν ἐστι τὸ ΑΒ



τοῦ Γ ἢ ὥστε ἰσορροπεῖν [τῷ Γ][1] ἢ οὔ. ἔστω μεῖζον, καὶ ἀφῃρήσθω ἀπὸ τοῦ ΑΒ ἔλασσον τᾶς ὑπεροχᾶς, ᾇ μεῖζόν ἐστι τὸ ΑΒ τοῦ Γ ἢ ὥστε ἰσορροπεῖν, ὥστε [τὸ][2] λοιπὸν τὸ Α σύμμετρον

And since ΛΕ = ΓΔ and ΕΓ = ΔΚ, therefore ΛΓ = ΓΚ; so that the centre of gravity of the magnitude compounded of them all is the point Γ. Therefore if Α is placed at Ε and Β at Δ, they will balance about Γ.

Prop. 7

And now, if the magnitudes be incommensurable, they will likewise balance at distances reciprocally proportional to the magnitudes.

Let (Α + Β), Γ be incommensurable magnitudes,[a] and let ΔΕ, ΕΖ be distances, and let

$$(Α + Β) : Γ = ΕΔ : ΕΖ\,;$$

I say that the centre of gravity of the magnitude composed of both (Α + Β), Γ is Ε.

For if (Α + Β) placed at Ζ do not balance Γ placed at Δ, either (Α + Β) is too much greater than Γ to balance or less. Let it [first] be too much greater, and let there be subtracted from (Α + Β) a magnitude less than the excess by which (Α + Β) is too much greater than Γ to balance, so that the remainder Α is

[a] As becomes clear later in the proof, the first magnitude is regarded as made up of two parts—Α, which is commensurate with Γ and Β, which is not commensurate; if (Α + Β) is too big for equilibrium with Γ, then Β is so chosen that, when it is taken away, the remainder Α is still too big for equilibrium with Γ. Similarly if (Α + Β) is too small for equilibrium.

[1] τῷ Γ om. Eutocius.
[2] τὸ om. Eutocius.

εἶμεν τῷ Γ. ἐπεὶ οὖν σύμμετρά ἐστι τὰ Α, Γ μεγέθεα, καὶ ἐλάσσονα λόγον ἔχει τὸ Α ποτὶ τὸ Γ ἢ ἁ ΔΕ ποτὶ ΕΖ, οὐκ ἰσορροπησοῦντι τὰ Α, Γ ἀπὸ τῶν ΔΕ, ΕΖ μακέων, τεθέντος τοῦ μὲν Α ἐπὶ τῷ Ζ, τοῦ δὲ Γ ἐπὶ τῷ Δ. διὰ ταὐτὰ δ', οὐδ' εἰ τὸ Γ μεῖζόν ἐστιν ἢ ὥστε ἰσορροπεῖν τῷ ΑΒ.

(iii.) *Centre of Gravity of a Parallelogram*

Ibid., Props. 9 et 10, Archim. ed. Heiberg ii. 140. 16–144. 4

θ′

Παντὸς παραλληλογράμμου τὸ κέντρον τοῦ βάρεός ἐστιν ἐπὶ τᾶς εὐθείας τᾶς ἐπιζευγνυούσας τὰς διχοτομίας τᾶν κατ' ἐναντίον τοῦ παραλληλογράμμου πλευρᾶν.

Ἔστω παραλληλόγραμμον τὸ ΑΒΓΔ, ἐπὶ δὲ τὰν διχοτομίαν τᾶν ΑΒ, ΓΔ ἁ ΕΖ· φαμὶ δή, ὅτι τοῦ ΑΒΓΔ παραλληλογράμμου τὸ κέντρον τοῦ βάρεος ἐσσεῖται ἐπὶ τᾶς ΕΖ.

Μὴ γάρ, ἀλλ', εἰ δυνατόν, ἔστω τὸ Θ, καὶ ἄχθω παρὰ τὰν ΑΒ ἁ ΘΙ. τᾶς [δὲ][1] δὴ ΕΒ διχοτομουμένας αἰεὶ ἐσσεῖταί ποκα ἁ καταλειπομένα ἐλάσσων

[1] δὲ om. Heiberg.

[a] The proof is incomplete and obscure; it may be thus completed.

Since

$$A : \Gamma < \Delta E : EZ,$$

Δ will be depressed, which is impossible, since there has been taken away from (A + B) a magnitude less than the deduction necessary to produce equilibrium, so that Z remains depressed. Therefore (A + B) is not greater than the magnitude necessary to produce equilibrium; in the same way it can be proved not to be less; therefore it is equal.

commensurate with Γ. Then, since A, Γ are commensurable magnitudes, and

$$A : \Gamma < \Delta E : EZ,$$

A, Γ will not balance at the distances ΔE, EZ, A being placed at Z and Γ at Δ. By the same reasoning, they will not do so if Γ is greater than the magnitude necessary to balance (A + B).[a]

(iii.) *Centre of Gravity of a Parallelogram*[b]

Ibid., Props. 9 and 10, Archim. ed. Heiberg ii. 140. 16–144. 4

Prop. 9

The centre of gravity of any parallelogram is on the straight line joining the points of bisection of opposite sides of the parallelogram.

Let ΑΒΓΔ be a parallelogram, and let EZ be the straight line joining the mid-points of AB, ΓΔ; then I say that the centre of gravity of the parallelogram ΑΒΓΔ will be on EZ.

For if it be not, let it, if possible, be Θ, and let ΘI be drawn parallel to AB. Now if EB be bisected, and the half be bisected, and so on continually, there will be left some line less than IΘ; [let EK be less than

[b] The centres of gravity of a triangle and a trapezium are also found by Archimedes in the first book; the second book is wholly devoted to finding the centres of gravity of a parabolic segment and of a portion of it cut off by a parallel to the base.