(c) Solutions by Higher Curves

(i.) General

Simpl. in Cat. 7, ed. Kalbfleisch 192. 15-25

Ἔστιν δὲ τετραγωνισμὸς κύκλου, ὅταν τῷ δο-
θέντι κύκλῳ ἴσον τετράγωνον συστησώμεθα. τοῦτο
δὲ ᾿Αριστοτέλης μέν, ὡς ἔοικεν, οὔπω ἐγνώκει,
παρὰ δὲ τοῖς Πυθαγορείοις ηὑρῆσθαί φησιν ᾿Ιάμ-
βλιχος, " ὡς δῆλόν ἐστιν ἀπὸ τῶν Σέξτου τοῦ
Πυθαγορείου ἀποδείξεων, ὃς ἄνωθεν κατὰ διαδοχὴν
παρέλαβεν τὴν μέθοδον τῆς ἀποδείξεως. καὶ
ὕστερον δέ, φησίν, ᾿Αρχιμήδης διὰ τῆς Λυκομήδους[1]
γραμμῆς καὶ Νικομήδης διὰ τῆς ἰδίως τετρα-
γωνιζούσης καλουμένης καὶ ᾿Απολλώνιος διά τινος
γραμμῆς, ἣν αὐτὸς μὲν κοχλιοειδοῦς ἀδελφὴν προσ-
αγορεύει, ἡ αὐτὴ δέ ἐστιν τῇ Νικομήδους, καὶ
Κάρπος δὲ διά τινος γραμμῆς, ἣ ἁπλῶς ἐκ διπλῆς
κινήσεως καλεῖ, ἄλλοι τε πολλοὶ ποικίλως τὸ πρό-
βλημα κατεσκεύασαν," ὡς ᾿Ιάμβλιχος ἱστορεῖ.

[1] No meaning can be extracted from Λυκομήδους, which
is an otherwise unknown word. The correct reading is
probably ἑλικοειδοῦς, " spiral-shaped."

(c) Solutions by Higher Curves

(i.) General

Simplicius, Commentary on Aristotle's Categories 7,
ed. Kalbfleisch 192. 15-25

The circle is squared when we construct a square
equal to the given circle. Aristotle, it would
appear, did not know how to do this, but Iam-
blichus says it was discovered by the Pythagoreans,
" as is plain from the proofs of Sextus the Pytha-
gorean,[a] who received the method of the proof from
early tradition. And later (he says), Archimedes
effected it by means of the spiral-shaped curve,[b]
Nicomedes by means of the curve known by the
special name quadratrix, Apollonius by means of a
certain curve which he himself calls sister of the
cochloid, but which is the same as Nicomedes' curve,[c]
Carpus by means of a certain curve which he simply
calls that arising from a double motion,[d] and many
others constructed a solution of this problem in
divers ways," as Iamblichus relates.

[a] Sextus (more properly Sextius) lived in the reign of
Augustus (or Tiberius) and there is no valid reason for
believing the early Pythagoreans solved the problem.

[b] Archimedes himself in his book On Spirals, which will
be noticed when we come to him, merely uses the spiral to
rectify the circle (Prop. 19). But the quadrature follows
from Measurement of a Circle, Prop. 1.

[c] Nothing further is known of Apollonius's " sister of the
cochloid," but Heath (H.G.M. i. 232) points out that
Apollonius wrote a treatise on the cochlias, or cylindrical
helix, that the subtangent to this curve can be used to square
the circular section of the cylinder, and that the name is
sufficiently akin to justify Apollonius in speaking of it as
the " sister of the cochloid."

[d] Tannery thought this was the cycloid, but there is no
evidence.

(ii.) *The Quadratrix*

Papp. *Coll.* iv. 30. 45–32. 50, ed. Hultsch 250. 33–258. 19

Construction of the Curve

λ΄. Εἰς τὸν τετραγωνισμὸν τοῦ κύκλου παρελήφθη τις ὑπὸ Δεινοστράτου καὶ Νικομήδους γραμμὴ καί τινων ἄλλων νεωτέρων ἀπὸ τοῦ περὶ αὐτὴν συμπτώματος λαβοῦσα τοὔνομα· καλεῖται γὰρ ὑπ᾽ αὐτῶν τετραγωνίζουσα καὶ γένεσιν ἔχει τοιαύτην.

Ἐκκείσθω τετράγωνον τὸ ΑΒΓΔ καὶ περὶ κέντρον τὸ Α περιφέρεια γεγράφθω ἡ ΒΕΔ, καὶ
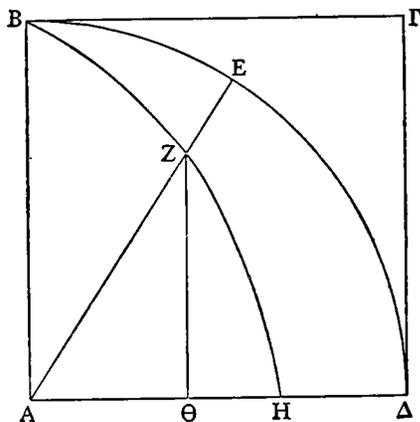
κινείσθω ἡ μὲν ΑΒ οὕτως ὥστε τὸ μὲν Α σημεῖον μένειν τὸ δὲ Β φέρεσθαι κατὰ τὴν ΒΕΔ περιφέρειαν, ἡ δὲ ΒΓ παράλληλος ἀεὶ διαμένουσα τῇ ΑΔ τῷ Β σημείῳ φερομένῳ[1] κατὰ τῆς ΒΑ συνακολουθείτω, καὶ ἐν ἴσῳ χρόνῳ ἥ τε ΑΒ κινουμένη

(ii.) *The Quadratrix*

Pappus, *Collection* iv. 30. 45–32. 50, ed. Hultsch 250. 33–258. 19

Construction of the Curve

30. For the squaring of the circle a certain line was used by Dinostratus and Nicomedes and certain other more recent geometers, and it takes its name from its special property ; for it is called by them the quadratrix,[a] and it is generated in this way.

Let ΑΒΓΔ be a square, and with centre A let the arc ΒΕΔ be described, and let AB be so moved that the point A remains fixed while B is carried along the arc ΒΕΔ ; furthermore let ΒΓ, while always remaining parallel to ΑΔ, follow the point B in its motion along BA, and in equal times let AB, moving uni-

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[a] Heath (*H.G.M.* i. 225-226) shows that the quadratrix was discovered by Hippias and that he may himself have used it (though this is not absolutely certain) to rectify, and so to square, the circle. He identifies Hippias with Hippias of Elis (*v. supra*, p. 149 n.*a*.)

[1] σημεῖον φέρον ἐν ᾧ in the MSS. was corrected by Torelli.

ὁμαλῶς τὴν ὑπὸ ΒΑΔ γωνίαν, τουτέστιν τὸ Β
σημεῖον τὴν ΒΕΔ περιφέρειαν, διαννέτω, καὶ ἡ
ΒΓ τὴν ΒΑ εὐθεῖαν παροδευέτω, τουτέστιν τὸ Β
σημεῖον κατὰ τῆς ΒΑ φερέσθω. συμβήσεται δῆλον
τῇ ΑΔ εὐθείᾳ ἅμα ἐφαρμόζειν ἑκατέραν τήν τε
ΑΒ καὶ τὴν ΒΓ. τοιαύτης δὴ γινομένης κινήσεως
τεμοῦσιν ἀλλήλας ἐν τῇ φορᾷ αἱ ΒΓ, ΒΑ εὐθεῖαι
κατά τι σημεῖον αἰεὶ συμμεθιστάμενον αὐταῖς, ὑφ'
οὗ σημείου γράφεταί τις ἐν τῷ μεταξὺ τόπῳ τῶν
τε ΒΑΔ εὐθειῶν καὶ τῆς ΒΕΔ περιφερείας γραμμὴ
ἐπὶ τὰ αὐτὰ κοίλη, οἵα ἐστὶν ἡ ΒΖΗ, ἣ καὶ χρειώ-
δης εἶναι δοκεῖ πρὸς τὸ τῷ δοθέντι κύκλῳ τετρά-
γωνον ἴσον εὑρεῖν. τὸ δὲ ἀρχικὸν αὐτῆς σύμπτωμα
τοιοῦτόν ἐστιν. ἥτις γὰρ ἂν διαχθῇ τυχοῦσα
⟨πρὸς τὴν περιφέρειαν, ὡς ἡ ΑΖΕ, ἔσται ὡς ὅλη
ἡ⟩[1] περιφέρεια πρὸς τὴν ΕΔ, ἡ ΒΑ εὐθεῖα πρὸς
τὴν ΖΘ· τοῦτο γὰρ ἐκ τῆς γενέσεως τῆς γραμμῆς
φανερόν ἐστιν.

Sporus's Criticisms

λα΄. Δυσαρεστεῖται δὲ αὐτῇ ὁ Σπόρος εὐλόγως
διὰ ταῦτα. πρῶτον μὲν γὰρ πρὸς ὃ δοκεῖ χρειώδης
εἶναι πρᾶγμα, τοῦτ' ἐν ὑποθέσει λαμβάνει. πῶς
γὰρ δυνατόν, δύο σημείων ἀρξαμένων ἀπὸ τοῦ Β

[1] πρὸς τὴν . . . ὅλη ἡ add. Hultsch.

formly, pass through the angle BAΔ (that is, the
point B pass along the arc BΔ), and BΓ pass by the
straight line BA (that is, let the point B traverse
the length of BA). Plainly then both AB and BΓ
will coincide simultaneously with the straight line
AΔ. While the motion is in progress the straight
lines BΓ, BA will cut one another in their movement
at a certain point which continually changes place
with them, and by this point there is described in
the space between the straight lines BA, AΔ and the
arc BEΔ a concave curve, such as BZH, which appears
to be serviceable for the discovery of a square equal
to the given circle. Its principal property is this.
If any straight line, such as AZE, be drawn to the
circumference, the ratio of the whole arc to EΔ will
be the same as the ratio of the straight line BA to
ZΘ ; for this is clear from the manner in which the
line was generated.[a]

Sporus's Criticisms [b]

31. With this Sporus is rightly displeased for these
reasons. In the first place, the end for which the
construction seems to be useful is assumed in the
hypothesis. For how is it possible, with two points

[a] If $AZ = \rho$, $\angle ZA\Delta = \phi$, $AB = a$, then the equation of the
curve is

$$\frac{\frac{1}{2}\pi}{\phi} = \frac{a}{\rho \sin \phi}$$

or

$$\pi\rho \sin \phi = 2a\phi.$$

[b] These acute criticisms of the quadratrix as a practical
method of squaring the circle appear to be well founded.
Sporus, who was not much older than Pappus himself,
lived towards the end of the third century A.D. He compiled
a work called Κηρία giving extracts on the quadrature of the
circle and duplication of the cube.

κινεῖσθαι, τὸ μὲν κατ᾽ εὐθείας ἐπὶ τὸ Α, τὸ δὲ
κατὰ περιφερείας ἐπὶ τὸ Δ ἐν ἴσῳ χρόνῳ συν-
αποκαταστῆσαι¹ μὴ πρότερον τὸν λόγον τῆς ΑΒ
εὐθείας πρὸς τὴν ΒΕΔ περιφέρειαν ἐπιστάμενον;
ἐν γὰρ τούτῳ τῷ λόγῳ καὶ τὰ τάχη τῶν κινήσεων
ἀνάγκη εἶναι. ἐπεὶ πῶς οἷόν τε συναποκαταστῆναι
τάχεσιν ἀκρίτοις χρώμενα, πλὴν εἰ μὴ ἂν κατὰ
τύχην ποτὲ συμβῇ; τοῦτο δὲ πῶς οὐκ ἄλογον;
ἔπειτα δὲ τὸ πέρας αὐτῆς ᾧ χρῶνται πρὸς τὸν
τετραγωνισμὸν τοῦ κύκλου, τουτέστιν καθ᾽ ὃ
τέμνει σημεῖον τὴν ΑΔ εὐθεῖαν, οὐχ εὑρίσκεται.
νοείσθω δὲ ἐπὶ τῆς προκειμένης τὰ λεγόμενα
καταγραφῆς· ὁπόταν γὰρ αἱ ΓΒ, ΒΑ φερόμεναι
συναποκατασταθῶσιν, ἐφαρμόσουσιν τῇ ΑΔ καὶ
τομὴν οὐκέτι ποιήσουσιν ἐν ἀλλήλαις· παύεται
γὰρ ἡ τομὴ πρὸ τῆς ἐπὶ τὴν ΑΔ ἐφαρμογῆς ἥπερ
τομὴ πέρας αὖ ἐγένετο τῆς γραμμῆς, καθ᾽ ὃ τῇ
ΑΔ εὐθείᾳ συνέπιπτεν. πλὴν εἰ μὴ λέγοι τις
ἐπινοεῖσθαι προσεκβαλλομένην τὴν γραμμήν, ὡς
ὑποτιθέμεθα τὰς εὐθείας, ἕως τῆς ΑΔ. τοῦτο δ᾽
οὐχ ἕπεται ταῖς ὑποκειμέναις ἀρχαῖς, ἀλλ᾽ ὡς ἂν
ληφθείη τὸ Η σημεῖον προειλημμένου τοῦ τῆς
περιφερείας πρὸς τὴν εὐθεῖαν λόγου. χωρὶς δὲ τοῦ
δοθῆναι τὸν λόγον τοῦτον οὐ χρὴ τῇ τῶν εὑρόντων
ἀνδρῶν δόξῃ πιστεύοντας παραδέχεσθαι τὴν γραμ-
μὴν μηχανικωτέραν πως οὖσαν [καὶ εἰς πολλὰ
προβλήματα χρησιμεύουσαν τοῖς μηχανικοῖς].²
ἀλλὰ πρότερον παραδεκτέον ἐστὶ τὸ δι᾽ αὐτῆς
δεικνύμενον πρόβλημα.

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¹ συναποκαταστῆναι coniecit Hultsch.
² καὶ . . . μηχανικοῖς interpolatori tribuit Hultsch.

beginning to move from B, to make one of them move along a straight line to A and the other along a circumference to Δ in equal time unless first the ratio of the straight line AB to the circumference BEΔ is known ? For it is necessary that the speeds of the moving points should be in this ratio. And how then could one, using unadjusted speeds, make the motions end together, unless this should sometimes happen by chance ? But how could this fail to be irrational ? Again, the extremity of the curve which they use for the squaring of the circle, that is, the point in which the curve cuts the straight line AΔ, is not found. Let the construction be conceived as aforesaid. When the straight lines ΓB, BA move so as to end their motion together, they will coincide with AΔ and will no longer cut each other. In fact, the inter-section ceases before the coincidence with AΔ, yet it was this intersection which was the extremity of the curve where it met the straight line AΔ. Unless, indeed, anyone should say the curve is conceived as produced, in the same way that we produce straight lines, as far as AΔ. But this does not follow from the assumptions made ; the point H can be found only by assuming the ratio of the circumference to the straight line. So unless this ratio is given, we must beware lest, in following the authority of those men who discovered the line, we admit its construction, which is more a matter of mechanics. But first let us deal with that problem which we have said can be proved by means of it.

Application of Quadratrix to Squaring of Circle

Τετραγώνου γὰρ ὄντος τοῦ ΑΒΓΔ καὶ τῆς μὲν
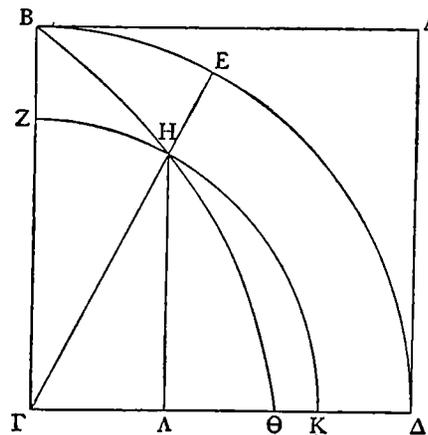περὶ τὸ κέντρον τὸ Γ περιφερείας τῆς ΒΕΔ, τῆς
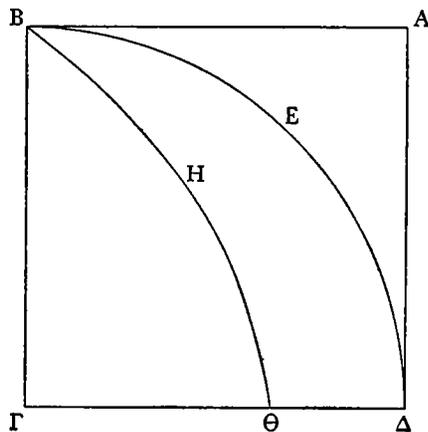
δὲ ΒΗΘ τετραγωνιζούσης γινομένης, ὡς προείρη-
ται, δείκνυται, ὡς ἡ ΔΕΒ περιφέρεια πρὸς τὴν
ΒΓ εὐθεῖαν, οὕτως ἡ ΒΓ πρὸς τὴν ΓΘ εὐθεῖαν.
εἰ γὰρ μὴ ἔστιν, ἤτοι πρὸς μείζονα ἔσται τῆς ΓΘ
ἢ πρὸς ἐλάσσονα.
Ἔστω πρότερον, εἰ δυνατόν, πρὸς μείζονα τὴν
ΓΚ, καὶ περὶ κέντρον τὸ Γ περιφέρεια ἡ ΖΗΚ
γεγράφθω τέμνουσα τὴν γραμμὴν κατὰ τὸ Η,
καὶ κάθετος ἡ ΗΛ, καὶ ἐπιζευχθεῖσα ἡ ΓΗ ἐκ-
βεβλήσθω ἐπὶ τὸ Ε. ἐπεὶ οὖν ἐστιν ὡς ἡ ΔΕΒ
περιφέρεια πρὸς τὴν ΒΓ εὐθεῖαν, οὕτως ἡ ΒΓ,

## Application of Quadratrix to Squaring of Circle

If ΑΒΓΔ is a square and ΒΕΔ the arc of a circle
with centre Γ, while ΒΗΘ is a quadratrix generated
in the aforesaid manner, it is proved that the ratio of
the arc ΔΕΒ towards the straight line ΒΓ is the same
as that of ΒΓ towards the straight line ΓΘ. For if it
is not, the ratio of the arc ΔΕΒ towards the straight
line ΒΓ will be the same as that of ΒΓ towards either
a straight line greater than ΓΘ or a straight line less
than ΓΘ.

Let it be the former, if possible, towards a greater
straight line ΓΚ, and with centre Γ let the arc ΖΗΚ
be drawn cutting the curve at Η, and let the perpen-
dicular ΗΛ be drawn, and let ΓΗ be joined and pro-

duced to Ε. Since therefore the ratio of the arc
ΔΕΒ towards the straight line ΒΓ is the same as the

τουτέστιν ἡ ΓΔ, πρὸς τὴν ΓΚ, ὡς δὲ ἡ ΓΔ πρὸς
τὴν ΓΚ, ἡ ΒΕΔ περιφέρεια πρὸς τὴν ΖΗΚ περι-
φέρειαν (ὡς γὰρ ἡ διάμετρος τοῦ κύκλου πρὸς τὴν
διάμετρον, ἡ περιφέρεια τοῦ κύκλου πρὸς τὴν
περιφέρειαν), φανερὸν ὅτι ἴση ἐστὶν ἡ ΖΗΚ περι-
φέρεια τῇ ΒΓ εὐθείᾳ. καὶ ἐπειδὴ διὰ τὸ σύμ-
πτωμα τῆς γραμμῆς ἐστιν ὡς ἡ ΒΕΔ περιφέρεια
πρὸς τὴν ΕΔ, οὕτως ἡ ΒΓ πρὸς τὴν ΗΛ, καὶ ὡς
ἄρα ἡ ΖΗΚ πρὸς τὴν ΗΚ περιφέρειαν, οὕτως
ἡ ΒΓ εὐθεῖα πρὸς τὴν ΗΛ. καὶ ἐδείχθη ἴση ἡ
ΖΗΚ περιφέρεια τῇ ΒΓ εὐθείᾳ· ἴση ἄρα καὶ ἡ
ΗΚ περιφέρεια τῇ ΗΛ εὐθείᾳ, ὅπερ ἄτοπον. οὐκ
ἄρα ἐστὶν ὡς ἡ ΒΕΔ περιφέρεια πρὸς τὴν ΒΓ
εὐθεῖαν, οὕτως ἡ ΒΓ πρὸς μείζονα τῆς ΓΘ.

λβ′. Λέγω δὲ ὅτι οὐδὲ πρὸς ἐλάσσονα. εἰ γὰρ
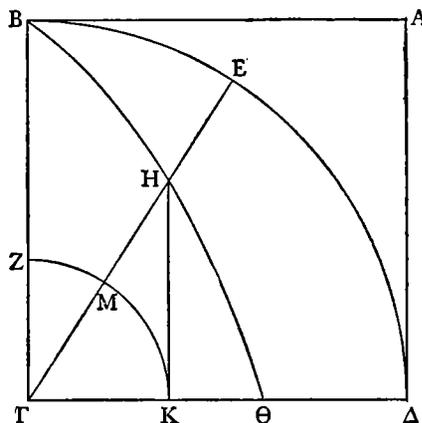
δυνατόν, ἔστω πρὸς τὴν ΚΓ, καὶ περὶ κέντρον τὸ

344

ratio of ΒΓ, that is ΓΔ, towards ΓΚ, and the ratio of
ΓΔ towards ΓΚ is the same as that of the arc ΒΕΔ
towards the arc ΖΗΚ (for the arcs of circles are in the
same ratio as their diameters), it is clear that the arc
ΖΗΚ is equal to the straight line ΒΓ. And since by
the property of the curve the ratio of the arc ΒΕΔ
towards ΕΔ is the same as the ratio of ΒΓ towards
ΗΛ, therefore the ratio of ΖΗΚ towards the arc ΗΚ
is the same as the ratio of the straight line ΒΓ
towards ΗΛ. And the arc ΖΗΚ was proved equal
to the straight line ΒΓ; therefore the arc ΗΚ is
also equal to the straight line ΗΛ, which is absurd.
Therefore the ratio of the arc ΒΕΔ towards the
straight line ΒΓ is not the same as the ratio of ΒΓ
towards a straight line greater than ΓΘ.

32. I say that neither is it equal to the ratio of ΒΓ
towards a straight line less than ΓΘ. For, if it is
possible, let the ratio be towards ΚΓ, and with centre

345

Γ περιφέρεια γεγράφθω ἡ ZMK, καὶ πρὸς ὀρθὰς τῇ ΓΔ ἡ ΚΗ τέμνουσα τὴν τετραγωνίζουσαν κατὰ τὸ Η, καὶ ἐπιζευχθεῖσα ἡ ΓΗ ἐκβεβλήσθω ἐπὶ τὸ Ε. ὁμοίως δὴ τοῖς προγεγραμμένοις δείξομεν καὶ τὴν ZMK περιφέρειαν τῇ ΒΓ εὐθείᾳ ἴσην, καὶ ὡς τὴν ΒΕΔ περιφέρειαν πρὸς τὴν ΕΔ, τουτέστιν ὡς τὴν ZMK πρὸς τὴν ΜΚ, οὕτως τὴν ΒΓ εὐθεῖαν πρὸς τὴν ΗΚ. ἐξ ὧν φανερὸν ὅτι ἴση ἔσται ἡ ΜΚ περιφέρεια τῇ ΚΗ εὐθείᾳ, ὅπερ ἄτοπον. οὐκ ἄρα ἔσται ὡς ἡ ΒΕΔ περιφέρεια πρὸς τὴν ΒΓ εὐθεῖαν, οὕτως ἡ ΒΓ πρὸς ἐλάσσονα τῆς ΓΘ. ἐδείχθη δὲ ὅτι οὐδὲ πρὸς μείζονα· πρὸς αὐτὴν ἄρα τὴν ΓΘ.

Ἔστι δὲ καὶ τοῦτο φανερὸν ὅτι ἡ τῶν ΘΓ, ΓΒ εὐθειῶν τρίτη ἀνάλογον λαμβανομένη εὐθεῖα ἴση ἔσται τῇ ΒΕΔ περιφερείᾳ, καὶ ἡ τετραπλασίων αὐτῆς τῇ τοῦ ὅλου κύκλου περιφερείᾳ. εὑρημένης δὲ τῇ τοῦ κύκλου περιφερείᾳ ἴσης εὐθείας πρόδηλον ὡς δὴ καὶ αὐτῷ τῷ κύκλῳ ῥᾴδιον ἴσον τετράγωνον συστήσασθαι· τὸ γὰρ ὑπὸ τῆς περιμέτρου τοῦ κύκλου καὶ τῆς ἐκ τοῦ κέντρου διπλάσιόν ἐστι τοῦ κύκλου, ὡς Ἀρχιμήδης ἀπέδειξεν.

3. TRISECTION OF AN ANGLE

(a) Types of Geometrical Problems

Papp. *Coll.* iv. 36. 57-59, ed. Hultsch 270. 1-272. 14

λϛ'. Τὴν δοθεῖσαν γωνίαν εὐθύγραμμον εἰς τρία ἴσα τεμεῖν οἱ παλαιοὶ γεωμέτραι θελήσαντες ἠπόρησαν δι' αἰτίαν τοιαύτην. τρία γένη φαμὲν εἶναι

Γ let the arc ZMK be described, and let KH at right angles to ΓΔ cut the quadratrix at H, and let ΓH be joined and produced to E. In similar manner to what has been written above, we shall prove also that the arc ZMK is equal to the straight line ΒΓ, and that the ratio of the arc ΒΕΔ towards ΕΔ, that is, the ratio of ZMK towards MK, is the same as that of the straight line ΒΓ towards HK. From this it is clear that the arc MK is equal to the straight line KH, which is absurd. The ratio of the arc ΒΕΔ towards the straight line ΒΓ is therefore not the same as the ratio of ΒΓ towards a straight line less than ΓΘ. Moreover it was proved not the same as the ratio of ΒΓ towards a straight line greater than ΓΘ ; therefore it is the same as the ratio of ΒΓ towards ΓΘ itself.

This also is clear, that if a straight line is taken as a third proportional to the straight lines ΘΓ, ΓΒ it will be equal to the arc ΒΕΔ, and four times this straight line will be equal to the circumference of the whole circle. A straight line equal to the circumference of the circle having been found, a square can easily be constructed equal to the circle itself. For the rectangle contained by the perimeter of the circle and the radius is double of the circle, as Archimedes demonstrated.[a]

3. TRISECTION OF AN ANGLE

(a) Types of Geometrical Problems

Pappus, *Collection* iv. 36. 57-59, ed. Hultsch 270. 1-272. 14

36. When the ancient geometers sought to divide a given rectilineal angle into three equal parts they were at a loss for this reason. We say that there

[a] See *supra*, pp. 316-321.