SELECTIONS
ILLUSTRATING THE HISTORY OF

GREEK MATHEMATICS

WITH AN ENGLISH TRANSLATION BY

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I

FROM THALES TO EUCLID

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IX. SPECIAL PROBLEMS

1. DUPLICATION OF THE CUBE

(a) General

Theon Smyr., ed. Hiller 2. 3-12

Ἐρατοσθένης μὲν γὰρ ἐν τῷ ἐπιγραφομένῳ Πλατωνικῷ φησιν ὅτι, Δηλίοις τοῦ θεοῦ χρήσαντος ἐπὶ ἀπαλλαγῇ λοιμοῦ βωμὸν τοῦ ὄντος διπλασίονα κατασκευάσαι, πολλὴν ἀρχιτέκτοσιν ἐμπεσεῖν ἀπορίαν ζητοῦσιν ὅπως χρὴ στερεὸν στερεοῦ γενέσθαι διπλάσιον, ἀφικέσθαι τε πευσομένους περὶ τούτου Πλάτωνος. τὸν δὲ φάναι αὐτοῖς, ὡς ἄρα οὐ διπλασίου βωμοῦ ὁ θεὸς δεόμενος τοῦτο Δηλίοις ἐμαντεύσατο, προφέρων δὲ καὶ ὀνειδίζων τοῖς Ἕλλησιν ἀμελοῦσι μαθημάτων καὶ γεωμετρίας ὠλιγωρηκόσιν.

Eutoc. *Comm. in Archim. de Sphaera et Cyl.* ii., Archim. ed. Heiberg iii. 88. 4–90. 13

Βασιλεῖ Πτολεμαίῳ Ἐρατοσθένης χαίρειν.

Τῶν ἀρχαίων τινὰ τραγῳδοποιῶν φασιν εἰσαγαγεῖν τὸν Μίνω τῷ Γλαύκῳ κατασκευάζοντα τάφον,

[a] Wilamowitz (*Gött. Nachr.*, 1894) shows that the letter is a forgery, but there is no reason to doubt the story it relates, which is indeed amply confirmed; and the author must be thanked for having included in his letter a proof and an

IX. SPECIAL PROBLEMS

1. DUPLICATION OF THE CUBE

(a) General

Theon of Smyrna, ed. Hiller 2. 3-12

In his work entitled *Platonicus* Eratosthenes says that, when the god announced to the Delians by oracle that to get rid of a plague they must construct an altar double of the existing one, their craftsmen fell into great perplexity in trying to find how a solid could be made double of another solid, and they went to ask Plato about it. He told them that the god had given this oracle, not because he wanted an altar of double the size, but because he wished, in setting this task before them, to reproach the Greeks for their neglect of mathematics and their contempt for geometry.

Eutocius, *Commentary on Archimedes' Sphere and Cylinder* ii., Archim. ed. Heiberg iii. 88. 4–90. 13

To King Ptolemy Eratosthenes sends greeting.[a]

They say that one of the ancient tragic poets represented Minos as preparing a tomb for Glaucus,

epigram, taken from a votive monument, which are the genuine work of Eratosthenes (*infra*, pp. 294-297). The monarch addressed is Ptolemy Euergetes, to whose son, Philopator, Eratosthenes was tutor.

πυθόμενον δέ, ὅτι πανταχοῦ ἑκατόμπεδος εἴη, εἰπεῖν·

μικρόν γ' ἔλεξας βασιλικοῦ σηκὸν τάφου·
διπλάσιος ἔστω, τοῦ καλοῦ δὲ μὴ σφαλεὶς
δίπλαζ' ἕκαστον κῶλον ἐν τάχει τάφου.

ἐδόκει δὲ διημαρτηκέναι· τῶν γὰρ πλευρῶν διπλασιασθεισῶν τὸ μὲν ἐπίπεδον γίνεται τετραπλάσιον, τὸ δὲ στερεὸν ὀκταπλάσιον. ἐζητεῖτο δὲ καὶ παρὰ τοῖς γεωμέτραις, τίνα ἄν τις τρόπον τὸ δοθὲν στερεὸν διαμένον ἐν τῷ αὐτῷ σχήματι διπλασιάσειεν, καὶ ἐκαλεῖτο τὸ τοιοῦτον πρόβλημα κύβου διπλασιασμός· ὑποθέμενοι γὰρ κύβον ἐζήτουν τοῦτον διπλασιάσαι. πάντων δὲ διαπορούντων ἐπὶ πολὺν χρόνον πρῶτος Ἱπποκράτης ὁ Χῖος ἐπενόησεν, ὅτι, ἐὰν εὑρεθῇ δύο εὐθειῶν γραμμῶν, ὧν ἡ μείζων τῆς ἐλάσσονός ἐστι διπλασία, δύο μέσας ἀνάλογον λαβεῖν ἐν συνεχεῖ ἀναλογίᾳ, διπλασιασθήσεται ὁ κύβος, ὥστε τὸ ἀπόρημα αὐτῷ εἰς ἕτερον οὐκ ἔλασσον ἀπόρημα κατέστρεφεν. μετὰ χρόνον δέ τινάς φασιν Δηλίους ἐπιβαλλομένους κατὰ χρησμὸν διπλασιάσαι τινὰ τῶν βωμῶν ἐμπεσεῖν εἰς τὸ αὐτὸ ἀπόρημα, διαπεμψαμένους δὲ τοὺς παρὰ τῷ Πλάτωνι ἐν Ἀκαδημίᾳ γεωμέτρας ἀξιοῦν αὐτοῖς εὑρεῖν τὸ ζητούμενον. τῶν δὲ φιλοπόνως ἐπιδιδόντων ἑαυτοὺς καὶ ζητούντων δύο τῶν δοθεισῶν δύο μέσας

[a] Valckenaer attributed these lines to Euripides, but Wilamowitz has shown that they cannot be from any play by Aeschylus, Sophocles or Euripides and must be the work of some minor poet.

[b] For if x, y are mean proportionals between a, b,

then $$\frac{a}{x}=\frac{x}{y}=\frac{y}{b}.$$

and as declaring, when he learnt it was a hundred feet each way: "Small indeed is the tomb thou hast chosen for a royal burial. Let it be double, and thou shalt not miss that fair form if thou quickly doublest each side of the tomb."[a] He seems to have made a mistake. For when the sides are doubled, the surface becomes four times as great and the solid eight times. It became a subject of inquiry among geometers in what manner one might double the given solid, while it remained the same shape, and this problem was called the duplication of the cube; for, given a cube, they sought to double it. When all were for a long time at a loss, Hippocrates of Chios first conceived that, if two mean proportionals could be found in continued proportion between two straight lines, of which the greater was double the lesser, the cube would be doubled,[b] so that the puzzle was by him turned into no less a puzzle. After a time, it is related, certain Delians, when attempting to double a certain altar in accordance with an oracle, fell into the same quandary, and sent over to ask the geometers who were with Plato in the Academy to find what they sought. When these men applied themselves diligently and sought to find two mean proportionals between two given straight lines,

Therefore $$y=\frac{x^2}{a}=\frac{ab}{x}$$

and, eliminating y, $$x^3=a^2b$$

so that $$\frac{a^3}{x^3}=\frac{a}{b}.$$

This property is stated in Eucl. *Elem.* v. Def. 10.
If $b=2a$, then x is the side of a cube double a cube of side a. Once this was discovered by Hippocrates, the problem was always so treated.

λαβεῖν Ἀρχύτας μὲν ὁ Ταραντῖνος λέγεται διὰ τῶν ἡμικυλίνδρων εὑρηκέναι, Εὔδοξος δὲ διὰ τῶν καλουμένων καμπύλων γραμμῶν· συμβέβηκε δὲ πᾶσιν αὐτοῖς ἀποδεικτικῶς γεγραφέναι, χειρουργῆσαι δὲ καὶ εἰς χρείαν πεσεῖν μὴ δύνασθαι πλὴν ἐπὶ βραχύ τι τὸν Μέναιχμον καὶ ταῦτα δυσχερῶς. ἐπινενόηται δέ τις ὑφ' ἡμῶν ὀργανικὴ λῆψις ῥᾳδία, δι' ἧς εὑρήσομεν δύο τῶν δοθεισῶν οὐ μόνον δύο μέσας, ἀλλ' ὅσας ἂν τις ἐπιτάξῃ.

(b) Solutions given by Eutocius

Eutoc. *Comm. in Archim. de Sphaera et Cyl.* ii., Archim. ed. Heiberg iii. 54. 26–56. 12

Εἰς τὴν σύνθεσιν τοῦ α'

Τούτου ληφθέντος ἐπεὶ δι' ἀναλύσεως αὐτῷ προέβη τὰ τοῦ προβλήματος, ληξάσης τῆς ἀναλύσεως εἰς τὸ δεῖν δύο δοθεισῶν δύο μέσας ἀνάλογον προσευρεῖν ἐν συνεχεῖ ἀναλογίᾳ φησὶν ἐν τῇ συνθέσει· "εὑρήσθωσαν." τὴν δὲ εὕρεσιν τούτων ὑπ' αὐτοῦ μὲν γεγραμμένην οὐδὲ ὅλως εὑρίσκομεν, πολλῶν δὲ κλεινῶν ἀνδρῶν γραφαῖς ἐντετυχήκαμεν τὸ πρόβλημα τοῦτο ἐπαγγελλομέναις, ὧν τὴν Εὐδόξου τοῦ Κνιδίου παρῃτησάμεθα γραφήν, ἐπειδή φησιν μὲν ἐν προοιμίοις διὰ καμπύλων γραμμῶν αὐτὴν ηὑρηκέναι, ἐν δὲ τῇ ἀποδείξει πρὸς τῷ μὴ κεχρῆσθαι καμπύλαις γραμμαῖς ἀλλὰ καὶ

[a] "Given a cone or cylinder, to find a sphere equal to the cone or cylinder" (Archim. ed. Heiberg i. 170-174).

[b] This is a great misfortune, as we may be sure Eudoxus would have treated the subject in his usual brilliant fashion.

Archytas of Taras is said to have found them by the half-cylinders, and Eudoxus by the so-called curved lines; but it turned out that all their solutions were theoretical, and they could not give a practical construction and turn it to use, except to a certain small extent Menaechmus, and that with difficulty. An easy mechanical solution was, however, found by me, and by means of it I will find, not only two means to the given straight lines, but as many as may be enjoined.

(b) Solutions given by Eutocius

Eutocius, *Commentary on Archimedes' Sphere and Cylinder* ii., Archim. ed. Heiberg iii. 54. 26–56. 12

On the Synthesis of Prop. 1 [a]

With this assumption the problem became for him one of analysis, and when the analysis resolved itself into the discovery of two mean proportionals in continuous proportion between two given straight lines he says in the synthesis: "Let them be found." How they were found we nowhere find described by him, but we have come across writings of many famous men dealing with this problem. Among them is Eudoxus of Cnidos, but we have omitted his account,[b] since he says in the preface that he made his discovery by means of curved lines, but in the demonstration itself not only did he not use curved

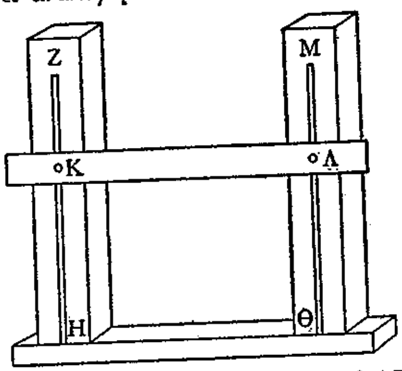
Tannery (*Mémoires scientifiques*, vol. i. pp. 53-61) suggests that Eudoxus's construction was a modified form of that by Archytas, for which see *infra*, pp. 284-289, the modification being virtually projection on the plane. Heath (*H.G.M.* i. 249-251) considers Tannery's suggestion ingenious and attractive, but too close an adaptation of Archytas's ideas to be the work or so original a mathematician as Eudoxus.

διῃρημένην ἀναλογίαν εὑρὼν ὡς συνεχεῖ χρῆται· ὅπερ ἦν ἄτοπον ὑπονοῆσαι, τί λέγω περὶ Εὐδόξου, ἀλλὰ περὶ τῶν καὶ μετρίως περὶ γεωμετρίαν ἀνεστραμμένων. ἵνα δὴ ἡ τῶν εἰς ἡμᾶς ἐληλυθότων ἀνδρῶν ἔννοια ἐμφανὴς γένηται, ὁ ἑκάστου τῆς εὑρέσεως τρόπος καὶ ἐνταῦθα γραφήσεται.

Ibid. 56. 13–58. 14

Ὡς Πλάτων

Δύο δοθεισῶν εὐθειῶν δύο μέσας ἀνάλογον εὑρεῖν ἐν συνεχεῖ ἀναλογίᾳ.



Ἔστωσαν αἱ δοθεῖσαι δύο εὐθεῖαι αἱ ΑΒΓ πρὸς

[a] The complete list of solutions given by Eutocius is: Plato, Heron, Philon, Apollonius, Diocles, Pappus, Sporus, Menaechmus (two solutions), Archytas, Eratosthenes, Nicomedes.

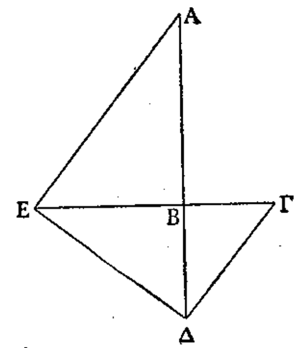
[b] It is virtually certain that this solution is wrongly attributed to Plato. Eutocius alone mentions it, and if it had been known to Eratosthenes he could hardly have failed to

lines but he used as continuous a discrete proportion which he found. That would be a foolish thing to imagine, not only of Eudoxus, but of any one moderately versed in geometry. In order that the ideas of those men who have come down to us may be made manifest, the manner in which each made his discovery will be described here also.[a]

Ibid. 56. 13–58. 14

(i.) *The Solution of Plato*[b]

Given two straight lines, to find two mean proportionals in continuous proportion.



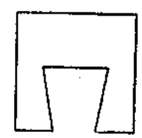
Let the two given straight lines be ΑΒ, ΒΓ, per-

cite it along with those of Archytas, Menaechmus and Eudoxus. Furthermore, Plato told the Delians, according to Plutarch's account, that Eudoxus or Helicon of Cyzicus would solve the problem for them: he did not apparently propose to tackle it himself. And Plutarch twice says that Plato objected to mechanical solutions as destroying the good of geometry, a statement which is consistent with his known attitude towards mathematics.

ὀρθὰς ἀλλήλαις, ὧν δεῖ δύο μέσας ἀνάλογον εὑρεῖν, ἐκβεβλήσθωσαν ἐπ' εὐθείας ἐπὶ τὰ Δ, Ε, καὶ κατεσκευάσθω ὀρθὴ γωνία ἡ ὑπὸ ΖΗΘ, καὶ ἐν ἑνὶ σκέλει, οἷον τῷ ΖΗ, κινείσθω κανὼν ὁ ΚΛ ἐν σωλῆνί τινι ὄντι ἐν τῷ ΖΗ οὕτως, ὥστε παράλληλον αὐτὸν διαμένειν τῷ ΗΘ. ἔσται δὲ τοῦτο, ἐὰν καὶ ἕτερον κανόνιον νοηθῇ συμφυὲς τῷ ΘΗ, παράλληλον δὲ τῷ ΖΗ, ὡς τὸ ΘΜ. σωληνισθεισῶν γὰρ τῶν ἄνωθεν ἐπιφανειῶν τῶν ΖΗ, ΘΜ σωλῆσιν πελεκινοειδέσιν καὶ τύλων συμφυῶν γενομένων τῷ ΚΛ εἰς τοὺς εἰρημένους σωλῆνας ἔσται ἡ κίνησις τοῦ ΚΛ παράλληλος ἀεὶ τῷ ΗΘ. τούτων οὖν κατεσκευασμένων κείσθω τὸ ἓν σκέλος τῆς γωνίας τυχὸν τὸ ΗΘ ψαῦον τοῦ Γ, καὶ μεταφερέσθω ἥ τε γωνία καὶ ὁ ΚΛ κανὼν ἐπὶ τοσοῦτον, ἄχρις ἂν τὸ μὲν Η σημεῖον ἐπὶ τῆς ΒΔ εὐθείας ᾖ τοῦ ΗΘ σκέλους ψαύοντος τοῦ Γ, ὁ δὲ ΚΛ κανὼν κατὰ μὲν τὸ Κ ψαύῃ τῆς ΒΕ εὐθείας, κατὰ δὲ τὸ λοιπὸν μέρος τοῦ Α, ὥστε εἶναι, ὡς ἔχει ἐπὶ τῆς καταγραφῆς, τὴν μὲν ὀρθὴν γωνίαν θέσιν ἔχουσαν ὡς τὴν ὑπὸ

pendicular to each other, between which it is required to find two mean proportionals. Let them be produced in a straight line to Δ, Ε, let the right-angle ΖΗΘ be constructed, and in one leg, say ΖΗ, let the ruler ΚΛ be moved in a kind of groove in ΖΗ, in such a way that it remains parallel to ΗΘ. This will come about if another ruler be conceived fixed to ΘΗ, but parallel to ΖΗ, such as ΘΜ. If the upper surfaces of ΖΗ, ΘΜ are grooved with axe-like grooves,[a] and there are notches on ΚΛ fitting into the aforementioned grooves, the motion of ΚΛ will always be parallel to ΗΘ. When this instrument is constructed, let one leg of the angle, say ΗΘ, be placed so as to touch Γ, and let the angle and the ruler ΚΛ be turned about until the point Η falls upon the straight line ΒΔ, while the leg ΗΘ touches Γ, and the ruler ΚΛ touches the straight line ΒΕ at Κ, and in the other part touches Α, so that it comes about, as in the figure, that the right angle takes up the position of the angle ΓΔΕ, while

[a] The grooves are presumably after the manner of the



accompanying diagram, or, as we should say, the notches and the grooves are *dove-tailed*.

ΓΔΕ, τὸν δὲ ΚΛ κανόνα θέσιν ἔχειν, οἵαν ἔχει ἡ ΕΑ· τούτων γὰρ γεναμένων ἔσται τὸ προκείμενον. ὀρθῶν γὰρ οὐσῶν τῶν πρὸς τοῖς Δ, Ε ἔστιν, ὡς ἡ ΓΒ πρὸς ΒΔ, ἡ ΔΒ πρὸς ΒΕ καὶ ἡ ΕΒ πρὸς ΒΑ.

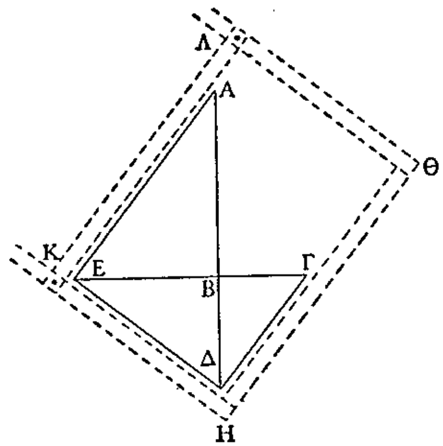
Ibid. 58. 15-16

Ὡς Ἥρων ἐν Μηχανικαῖς εἰσαγωγαῖς καὶ ἐν τοῖς Βελοποιικοῖς

Papp. *Coll.* iii. 9. 26, ed. Hultsch 62. 19–64. 18; Heron, *Mech.* i. 11, ed. Schmidt 268. 3–270. 15

Ἔστωσαν γὰρ αἱ δοθεῖσαι εὐθεῖαι αἱ ΑΒ, ΒΓ πρὸς ὀρθὰς ἀλλήλαις κείμεναι, ὧν δεῖ δύο μέσας ἀνάλογον εὑρεῖν.

[a] The account may become clearer from the accompanying diagram in which the instrument is indicated in its final



the ruler ΚΛ takes up the position ΕΑ.[a] When this is done, what was enjoined will be brought about. For since the angles at Δ, Ε are right, ΓΒ : ΒΔ = ΔΒ : ΒΕ = ΕΒ : ΒΑ. [Eucl. vi. 8, coroll.]

Ibid. 58. 15-16

(ii.) *The Solution of Heron in his "Mechanics" and "Construction of Engines of War"*[b]

Pappus, *Collection* iii. 9. 26, ed. Hultsch 62. 19–64. 18; Heron, *Mechanics* i. 11, ed. Schmidt 268. 3–270. 15

Let the two given straight lines between which it is required to find two mean proportionals be ΑΒ, ΒΓ lying at right angles one to another.

position by dotted lines. ΗΘ is made to pass through Γ and the instrument is turned until the point Η lies on ΑΒ produced. The ruler is then moved until its edge ΚΛ passes through Α. If Κ does not then lie on ΓΒ produced, the instrument has to be manipulated again until all conditions are fulfilled: (1) ΗΘ passes through Γ; (2) Η lies on ΑΒ produced; (3) ΚΛ passes through Α; (4) Κ lies on ΓΒ produced. It may not be easy to do this, but it is possible.

[b] Heron's own words have been most closely preserved by Pappus, whose version is here given in preference to Eutocius's, which includes some additions by the commentator. Schmidt also prefers Pappus's version in his edition of the Greek fragments of Heron's *Mechanics* in the Teubner edition of Heron's works (vol. ii., fasc. 1). The proof in the *Belopoeica* (edited by Wescher, *Poliorcétique des Grecs*, pp. 116-119) is extant. Philon of Byzantium and Apollonius gave substantially identical proofs.

Συμπεπληρώσθω τὸ ΑΒΓΔ παραλληλόγραμμον, καὶ ἐκβεβλήσθωσαν αἱ ΔΓ, ΔΑ, καὶ ἐπεζεύχθωσαν αἱ ΔΒ, ΓΑ, καὶ παρακείσθω κανόνιον πρὸς τῷ Β σημείῳ καὶ κινείσθω τέμνον τὰς ΓΕ, ΑΖ, ἄχρις οὗ ἡ ἀπὸ τοῦ Η ⟨ἀχθεῖσα⟩[1] ἐπὶ τὴν τῆς ΓΕ τομὴν ἴση γένηται τῇ ἀπὸ τοῦ Η ἐπὶ τὴν τῆς ΑΖ τομήν. γεγονέτω, καὶ ἔστω ἡ μὲν τοῦ κανονίου θέσις ἡ ΕΒΖ, ἴσαι δὲ αἱ ΕΗ, ΗΖ. λέγω οὖν ὅτι αἱ ΑΖ, ΓΕ μέσαι ἀνάλογόν εἰσιν τῶν ΑΒ, ΒΓ.

Ἐπεὶ γὰρ ὀρθογώνιόν ἐστιν τὸ ΑΒΓΔ παραλληλόγραμμον, αἱ τέσσαρες εὐθεῖαι αἱ ΔΗ, ΗΑ, ΗΒ, ΗΓ ἴσαι ἀλλήλαις εἰσίν. ἐπεὶ οὖν ἴση ἡ ΔΗ τῇ ΑΗ καὶ διῆκται ἡ ΗΖ, τὸ ἄρα ὑπὸ ΔΖΑ μετὰ τοῦ ἀπὸ ΑΗ ἴσον ἐστὶν τῷ ἀπὸ ΗΖ. διὰ τὰ αὐτὰ δὴ καὶ τὸ ὑπὸ ΔΕΓ μετὰ τοῦ ἀπὸ ΓΗ ἴσον ἐστὶν τῷ ἀπὸ ΗΕ. καὶ εἰσὶν ἴσαι αἱ ΗΕ,

[1] ἀχθεῖσα add. Hultsch.

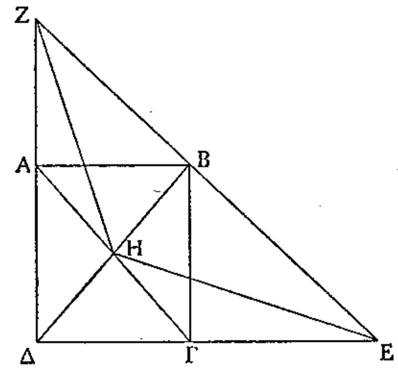
[a] The full proof requires ΗΘ to be drawn perpendicular to ΔΖ so that Θ bisects ΔΑ.

Then $ΔΖ . ΖΑ + ΑΘ^2 = ΖΘ^2$. [Eucl. ii. 6

Add $ΗΘ^2$ to each side.

Then $ΔΖ . ΖΑ + ΑΗ^2 = ΗΖ^2$. [Eucl. i. 47

Let the parallelogram ΑΒΓΔ be completed, and let ΔΓ, ΔΑ be produced and let ΔΒ, ΓΑ be joined,



and let a ruler be placed at Β and moved about until the sections ΓΕ, ΑΖ cut off [from ΔΓ, ΔΑ produced] are such that the straight line drawn from Η to the section ΓΕ is equal to the straight line drawn from Η to the section ΑΖ. Let this be done, and let the position of the ruler be ΕΒΖ, so that ΕΗ, ΗΖ are equal. I say that ΑΖ, ΓΕ are mean proportionals between ΑΒ, ΒΓ.

For since the parallelogram ΑΒΓΔ is right-angled, the four straight lines ΔΗ, ΗΑ, ΗΒ, ΗΓ are equal one to another. Since ΔΗ is equal to ΑΗ, and ΗΖ has been drawn (from the vertex of the isosceles triangle ΑΗΔ to the base), therefore [a]

$$ΔΖ . ΖΑ + ΑΗ^2 = ΗΖ^2.$$

For the same reasons

$$ΔΕ . ΕΓ + ΓΗ^2 = ΗΕ^2.$$

But ΗΕ, ΗΖ are equal.

ΗΖ. ἴσον ἄρα καὶ τὸ ὑπὸ ΔΖΑ μετὰ τοῦ ἀπὸ ΑΗ τῷ ὑπὸ ΔΕΓ μετὰ τοῦ ἀπὸ ΓΗ. ὧν τὸ ἀπὸ ΓΗ ἴσον ἐστὶν τῷ ἀπὸ ΗΑ. λοιπὸν ἄρα τὸ ὑπὸ ΔΕΓ ἴσον ἐστὶν τῷ ὑπὸ ΔΖΑ. ὡς ἄρα ἡ ΕΔ πρὸς ΔΖ, ἡ ΖΑ πρὸς ΓΕ. ὡς δε ἡ ΕΔ πρὸς ΔΖ, ἥ τε ΒΑ πρὸς ΑΖ καὶ ἡ ΕΓ πρὸς ΓΒ, ὥστε ἔσται καὶ ὡς ἡ ΑΒ πρὸς ΑΖ, ἥ τε ΖΑ πρὸς ΓΕ καὶ ἡ ΓΕ πρὸς ΓΒ. τῶν ἄρα ΑΒ, ΒΓ μέσαι ἀνάλογόν εἰσιν αἱ ΑΖ, ΓΕ.]

Eutoc. *Comm. in Archim. De Sphaera et Cyl.* ii., Archim. ed. Heiberg iii. 66. 8–70. 5

Ὡς Διοκλῆς ἐν τῷ Περὶ πυρίων

Ἐν κύκλῳ ἤχθωσαν δύο διάμετροι πρὸς ὀρθὰς αἱ ΑΒ, ΓΔ, καὶ δύο περιφέρειαι ἴσαι ἀπειλήφθωσαν ἐφ' ἑκάτερα τοῦ Β αἱ ΕΒ, ΒΖ, καὶ διὰ τοῦ Ζ παράλληλος τῇ ΑΒ ἤχθω ἡ ΖΗ, καὶ ἐπεζεύχθω ἡ ΔΕ. λέγω, ὅτι τῶν ΓΗ, ΗΘ δύο μέσαι ἀνάλογόν εἰσιν αἱ ΖΗ, ΗΔ.

Ἤχθω γὰρ διὰ τοῦ Ε τῇ ΑΒ παράλληλος ἡ

[a] The Greek text of the Περὶ πυρίων of Diocles has been lost but two fragments attributed to him by name but to some extent reformulated in contemporary mathematical language have been preserved by Eutocius—the one here translated and the other in Archim. ed. Heiberg iii. 160. 2–174. 4. This contains a solution by

Therefore $\Delta Z \,.\, ZA + AH^2 = \Delta E \,.\, E\Gamma + \Gamma H^2$.

And $AH^2 = \Gamma H^2$.

Therefore $\Delta Z \,.\, ZA = \Delta E \,.\, E\Gamma$.

Therefore $E\Delta : \Delta Z = ZA : \Gamma E$.

But (by similar triangles)

$$E\Delta : \Delta Z = BA : AZ = E\Gamma : \Gamma B,$$

so that $AB : AZ = ZA : \Gamma E = \Gamma E : \Gamma B$.

Therefore AZ, ΓE are mean proportionals between AB, BΓ.]

Eutocius, *Commentary on Archimedes' Sphere and Cylinder* ii., Archim. ed. Heiberg iii. 66. 8–70. 5

(iii.) *The Solution of Diocles in his Book "On Burning Mirrors"* [a]

In a circle let there be drawn two diameters AB, ΓΔ at right angles, and on either side of B let there be cut off two equal arcs EB, BZ, and through Z let ZH be drawn parallel to AB, and let ΔE be joined. I say that ZH, HΔ are two mean proportionals between ΓH, HΘ.

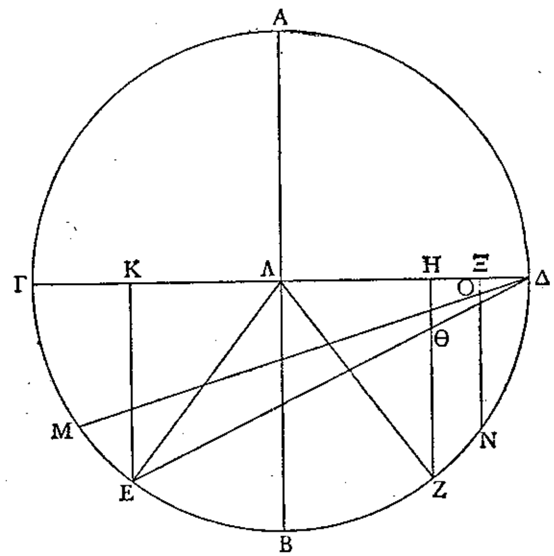
For let EK be drawn through E parallel to AB;

means of conics of the problem of dividing a sphere by a plane in such a way that the volumes of the resulting segments shall be in a given ratio. An Arabic translation of the whole work has, however, now been found in the Shrine Library at Meshed and has been edited by G. J. Toomer as *Diocles On Burning Mirrors*. The Arabic text shows, in Toomer's opinion, that another passage in Eutocius—ed. Heiberg iii. 82. 2–84. 7—should also be attributed to Diocles though he is not mentioned by name; as it follows the solution of finding two mean proportionals by Menaechmus and is introduced with the word Ἄλλως, "Otherwise," ΕΚ· ἴση ἄρα ἐστὶν ἡ μὲν ΕΚ τῇ ΖΗ, ἡ δὲ ΚΓ τῇ ΗΔ. ἔσται γὰρ τοῦτο δῆλον ἀπὸ τοῦ Λ ἐπὶ τὰ Ε, Ζ ἐπιζευχθεισῶν εὐθειῶν· ἴσαι γὰρ γίνονται αἱ ὑπὸ ΓΛΕ, ΖΛΔ, καὶ ὀρθαὶ αἱ πρὸς τοῖς Κ, Η· καὶ πάντα ἄρα πᾶσιν διὰ τὸ τὴν ΛΕ τῇ ΛΖ ἴσην εἶναι· καὶ λοιπὴ ἄρα ἡ ΓΚ τῇ ΗΔ ἴση ἐστίν. ἐπεὶ οὖν ἐστιν, ὡς ἡ ΔΚ πρὸς ΚΕ, ἡ ΔΗ πρὸς ΗΘ, ἀλλ' ὡς ἡ ΔΚ πρὸς ΚΕ, ἡ ΕΚ πρὸς ΚΓ· μέση γὰρ ἀνάλογον ἡ ΕΚ τῶν ΔΚ, ΚΓ· ὡς ἄρα ἡ ΔΚ πρὸς ΚΕ καὶ ἡ ΕΚ πρὸς ΚΓ, οὕτως ἡ ΔΗ

it has hitherto been attributed to that geometer.

It used to be thought from references to Archimedes and Apollonius in the second fragment mentioned above that Diocles lived after those geometers, but the references to Apollonius are clearly insertions by Eutocius. It used also to be argued from allusions in Proclus's commentary on Euclid, *Elements* i, that the curve known to Geminus as the cissoid (κισσοειδὴς (γραμμή), "ivy-shaped") was none other than the curve used by Diocles for finding two mean proportionals, but this identification does not ante-date the 17th century and is dubious. The Arabic translation has now enabled this disputed question of the date of Diocles to be settled, for it shows him to have been in contact with Zenodorus, who was himself an associate of the philosopher Philonides. This has enabled Toomer to assign the "floruit" of Diocles with assurance to the early 2nd century B.C.

EK will therefore be equal to ZH, and KΓ to HΔ; this will be clear if straight lines are drawn joining



Λ to E, Z; for the angles ΓΛE, ZΛΔ are equal, and the angles at K, H are right; and therefore, since ΛE = ΛZ, all things will be equal to all; and therefore the remaining element ΓK is equal to HΔ. Now since

$$\Delta K : KE = \Delta H : H\Theta,$$

but $\Delta K : KE = EK : K\Gamma$ (for EK is a mean proportional between ΔK, KΓ),

therefore $\Delta K : KE = EK : K\Gamma = \Delta H : H\Theta$.

πρὸς ΗΘ. καί ἐστιν ἴση ἡ μὲν ΔΚ τῇ ΓΗ, ἡ δὲ ΚΕ τῇ ΖΗ, ἡ δὲ ΚΓ τῇ ΗΔ· ὡς ἄρα ἡ ΓΗ πρὸς ΗΖ, ἡ ΖΗ πρὸς ΗΔ καὶ ἡ ΔΗ πρὸς ΗΘ. ἐὰν δὴ παρ' ἑκάτερα τοῦ Β ληφθῶσιν περιφέρειαι ἴσαι αἱ ΜΒ, ΒΝ, καὶ διὰ μὲν τοῦ Ν παράλληλος ἀχθῇ τῇ ΑΒ ἡ ΝΞ, ἐπιζευχθῇ δὲ ἡ ΔΜ, ἔσονται πάλιν τῶν ΓΞ, ΞΟ μέσαι ἀνάλογον αἱ ΝΞ, ΞΔ. πλειόνων οὖν οὕτως καὶ συνεχῶν παραλλήλων ἐκβληθεισῶν μεταξὺ τῶν Β, Δ καὶ ταῖς ἀπολαμβανομέναις ὑπ' αὐτῶν περιφερείαις πρὸς τῷ Β ἴσων τεθεισῶν ἀπὸ τοῦ Β ὡς ἐπὶ τὸ Γ καὶ ἐπὶ τὰ γενάμενα σημεῖα ἐπιζευχθεισῶν εὐθειῶν ἀπὸ τοῦ Δ, ὡς τῶν ὁμοίων ταῖς ΔΕ, ΔΜ, τμηθήσονται αἱ παράλληλοι αἱ μεταξὺ τῶν Β, Δ κατά τινα σημεῖα, ἐπὶ τῆς προκειμένης καταγραφῆς τὰ Ο, Θ, ἐφ' ἃ κανόνος παραθέσει ἐπιζεύξαντες εὐθείας ἕξομεν καταγε-

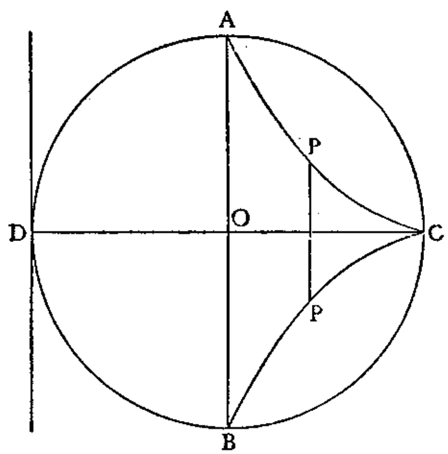
And $\Delta K = \Gamma H$, $KE = ZH$, $K\Gamma = H\Delta$;

therefore $\Gamma H : HZ = ZH : H\Delta = \Delta H : H\Theta$.

If then on either side of B there be cut off equal arcs MB, BN, and NΞ be drawn through N parallel to AB, and ΔM be joined, NΞ, ΞΔ, will again be mean proportionals between ΓΞ, ΞO. If in this way more parallels are drawn continually between B, Δ, and arcs equal to the arcs cut off between them and B are marked off from B in the direction of Γ, and straight lines are drawn from Δ to the points so obtained, such as ΔE, ΔM, the parallels between B and Δ will be cut in certain points, such as O, Θ in the accompanying figure. Joining these points with straight lines by applying a ruler we shall describe in the γραμμένην ἐν τῷ κύκλῳ τινὰ γραμμήν, ἐφ' ἧς ἐὰν ληφθῇ τυχὸν σημεῖον καὶ δι' αὐτοῦ παράλληλος ἀχθῇ τῇ ΛΒ, ἔσται ἡ ἀχθεῖσα καὶ ἡ ἀπολαμβανομένη ὑπ' αὐτῆς ἀπὸ τῆς διαμέτρου πρὸς τῷ Δ μέσαι ἀνάλογον τῆς τε ἀπολαμβανομένης ὑπ' αὐτῆς ἀπὸ τῆς διαμέτρου πρὸς τῷ Γ σημείῳ καὶ τοῦ μέρους αὐτῆς τοῦ ἀπὸ τοῦ ἐν τῇ γραμμῇ σημείου ἐπὶ τὴν ΓΔ διάμετρον.

Τούτων προκατεσκευασμένων ἔστωσαν αἱ δο-

[a] Lit. "line." It is noteworthy that Diocles, or Eutocius, conceived the curve as made up of an indefinite number of

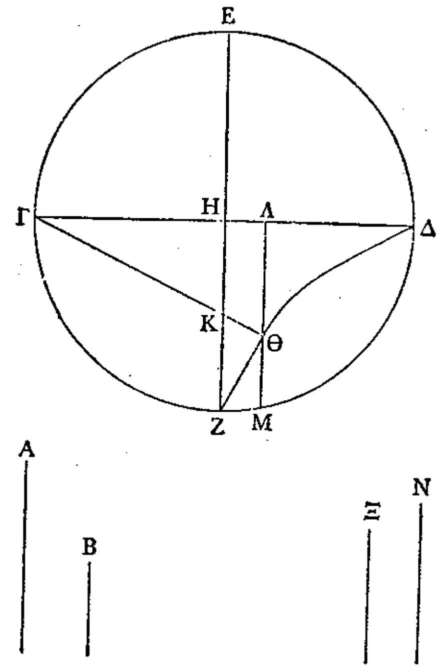


small straight lines, a typical Greek conception which has all the power of a theory of infinitesimals while avoiding its logical fallacies. The Greeks were never so modern as in this conception.

The curve described by Diocles has two branches, sym-

circle a certain curve,[a] and if on this any point be taken at random, and through it a straight line be drawn parallel to ΛΒ, the line so drawn and the portion of the diameter cut off by it in the direction of Δ will be mean proportionals between the portion of the diameter cut off by it in the direction of the point Γ and the part of the parallel itself between the point on the curve and the diameter ΓΔ.

With this preliminary construction, let the two



θεῖσαι δύο εὐθεῖαι, ὧν δεῖ δύο μέσας ἀνάλογον εὑρεῖν, αἱ Α, Β, καὶ ἔστω κύκλος, ἐν ᾧ δύο διάμετροι πρὸς ὀρθὰς ἀλλήλαις αἱ ΓΔ, ΕΖ, καὶ γεγράφθω ἐν αὐτῷ ἡ διὰ τῶν συνεχῶν σημείων γραμμή, ὡς προείρηται, ἡ ΔΘΖ, καὶ γεγονέτω, ὡς ἡ Α πρὸς τὴν Β, ἡ ΓΗ πρὸς ΗΚ, καὶ ἐπιζευχθεῖσα ἡ ΓΚ καὶ ἐκβληθεῖσα τεμνέτω τὴν γραμμὴν κατὰ τὸ Θ, καὶ διὰ τοῦ Θ τῇ ΕΖ παράλληλος ἤχθω ἡ ΛΜ· διὰ ἄρα τὰ προγεγραμμένα τῶν ΓΛ, ΛΘ μέσαι ἀνάλογόν εἰσιν αἱ ΜΛ, ΛΔ. καὶ ἐπεί ἐστιν, ὡς ἡ ΓΛ πρὸς ΛΘ, οὕτως ἡ ΓΗ πρὸς ΗΚ, ὡς δὲ ἡ ΓΗ πρὸς ΗΚ, οὕτως ἡ Α πρὸς τὴν Β, ἐὰν ἐν τῷ αὐτῷ λόγῳ ταῖς ΓΛ, ΛΜ, ΛΔ, ΛΘ παρεμβάλωμεν μέσας τῶν Α, Β, ὡς τὰς Ν, Ξ, ἔσονται εἰλημμέναι τῶν Α, Β μέσαι ἀνάλογον αἱ Ν, Ξ· ὅπερ ἔδει εὑρεῖν.

Ibid. 78. 13–80. 24

Ὡς Μέναιχμος

Ἔστωσαν αἱ δοθεῖσαι δύο εὐθεῖαι αἱ Α, Ε· δεῖ δὴ τῶν Α, Ε δύο μέσας ἀνάλογον εὑρεῖν.

Γεγονέτω, καὶ ἔστωσαν αἱ Β, Γ, καὶ ἐκκείσθω θέσει εὐθεῖα ἡ ΔΗ πεπερασμένη κατὰ τὸ Δ, καὶ πρὸς τῷ Δ τῇ Γ ἴση κείσθω ἡ ΔΖ, καὶ ἤχθω πρὸς ὀρθὰς ἡ ΖΘ, καὶ τῇ Β ἴση κείσθω ἡ ΖΘ. ἐπεὶ οὖν τρεῖς εὐθεῖαι ἀνάλογον αἱ Α, Β, Γ, τὸ ὑπὸ τῶν Α, Γ ἴσον ἐστὶ τῷ ἀπὸ τῆς Β· τὸ ἄρα ὑπὸ

metrical about the diameter CD in the accompanying figure, and proceeding to infinity. There is a cusp at C and the tangent to the circle at D is an asymptote. If OC is the axis of x, and OA the axis of y, while the radius of the circle is

given straight lines, between which it is required to find two mean proportionals, be A, B, and let there be a circle in which ΓΔ, EZ are two diameters at right angles to each other, and let there be drawn in it through the successive points a curve ΔΘZ, in the aforesaid manner, and let A : B = ΓH : HK, and let Γ, K be joined, and let the straight line joining them be produced so as to cut the line in Θ, and through Θ let ΛM be drawn parallel to EZ; therefore by what has been written previously MΛ, ΛΔ are mean proportionals between ΓΛ, ΛΘ. And since ΓΛ : ΛΘ = ΓH : HK and ΓH : HK = A : B, if between A, B we place means N, Ξ in the same ratio as ΓΛ, ΛM, ΛΔ, ΛΘ,[a] then N, Ξ will be mean proportionals between A, B; which was to be found.

Ibid. 78. 13–80. 24

(iv.) *The Solution of Menaechmus*

Let the two given straight lines be A, E; it is required to find two mean proportionals between A, E.

Assume it done, and let the means be B, Γ, and let there be placed in position a straight line ΔH, with an end point Δ, and at Δ let ΔZ be placed equal to Γ, and let ZΘ be drawn at right angles and let ZΘ be equal to B. Since the three straight lines A, B, Γ are in proportion, A.Γ = B²; therefore the rectangle com-

a, then by definition the Cartesian equation of the curve is

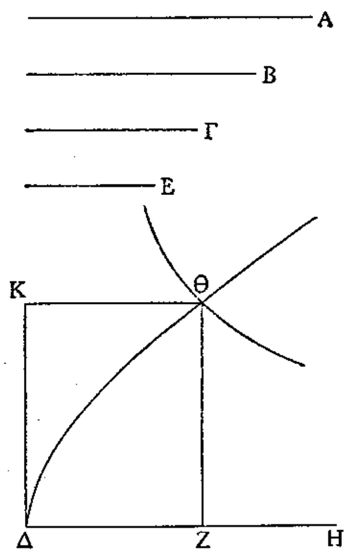
$$\frac{a+x}{\sqrt{a^2-x^2}}=\frac{a-x}{y} \text{ or } y^2(a+x)=(a-x)^3.$$

[a] *i.e.*, if we take ΓΛ : ΛM = A : N, ΛM : ΛΔ = N : Ξ and ΛΔ : ΛΘ = Ξ : B.

δοθείσης τῆς Α καὶ τῆς Γ, τουτέστι τῆς ΔΖ, ἴσον ἐστὶ τῷ ἀπὸ τῆς Β, τουτέστι τῷ ἀπὸ τῆς ΖΘ. ἐπὶ παραβολῆς ἄρα τὸ Θ διὰ τοῦ Δ γεγραμμένης. ἤχθωσαν παράλληλοι αἱ ΘΚ, ΔΚ. καὶ ἐπεὶ δοθὲν τὸ ὑπὸ Β, Γ—ἴσον γάρ ἐστι τῷ ὑπὸ Α, Ε—δοθὲν ἄρα καὶ τὸ ὑπὸ ΚΘΖ. ἐπὶ ὑπερβολῆς ἄρα τὸ Θ ἐν ἀσυμπτώτοις ταῖς ΚΔ, ΔΖ. δοθὲν ἄρα τὸ Θ· ὥστε καὶ τὸ Ζ.

Συντεθήσεται δὴ οὕτως. ἔστωσαν αἱ μὲν δοθεῖσαι εὐθεῖαι αἱ Α, Ε, ἡ δὲ τῇ θέσει ἡ ΔΗ πεπερασμένη κατὰ τὸ Δ, καὶ γεγράφθω διὰ τοῦ Δ

prehended by the given straight line A and the straight line Γ, that is, ΔZ, is equal to the square on



B, that is, to the square on ZΘ. Therefore Θ is on a parabola drawn through Δ. Let the parallels ΘK, ΔK be drawn. Then since the rectangle B . Γ is given —for it is equal to the rectangle A . E—the rectangle KΘ . ΘZ is given. The point Θ is therefore on a hyperbola with asymptotes KΔ, ΔZ. Therefore Θ is given; and so also is Z.

Let the synthesis be made in this manner. Let the given straight lines be A, E, let ΔH be a straight line given in position with an end point at Δ, and let παραβολή, ἧς ἄξων μὲν ἡ ΔΗ, ὀρθία δὲ τοῦ εἴδους πλευρὰ ἡ Α, αἱ δὲ καταγόμεναι ἐπὶ τὴν ΔΗ ἐν ὀρθῇ γωνίᾳ δυνάσθωσαν τὰ παρὰ τὴν Α παρακείμενα χωρία πλάτη ἔχοντα τὰς ἀπολαμβανομένας ὑπ' αὐτῶν πρὸς τῷ Δ σημείῳ. γεγράφθω καὶ ἔστω ἡ ΔΘ, καὶ ὀρθὴ ἡ ΔΚ, καὶ ἐν ἀσυμπτώτοις ταῖς ΚΔ, ΔΖ γεγράφθω ὑπερβολή, ἀφ' ἧς αἱ παρὰ τὰς ΚΔ, ΔΖ ἀχθεῖσαι ποιήσουσιν τὸ χωρίον ἴσον τῷ ὑπὸ Α, Ε· τεμεῖ δὴ τὴν παραβολήν. τεμνέτω κατὰ τὸ Θ, καὶ κάθετοι ἤχθωσαν αἱ ΘΚ, ΘΖ. ἐπεὶ οὖν τὸ ἀπὸ ΖΘ ἴσον ἐστὶ τῷ ὑπὸ Α, ΔΖ, ἔστιν, ὡς ἡ Α πρὸς τὴν ΖΘ, ἡ ΘΖ πρὸς ΖΔ. πάλιν, ἐπεὶ τὸ ὑπὸ Α, Ε ἴσον ἐστὶ τῷ ὑπὸ ΘΖΔ, ἔστιν, ὡς ἡ Α πρὸς τὴν ΖΘ, ἡ ΖΔ πρὸς τὴν Ε. ἀλλ' ὡς ἡ Α πρὸς τὴν ΖΘ, ἡ ΖΘ πρὸς ΖΔ· καὶ ὡς ἄρα ἡ Α πρὸς τὴν ΖΘ, ἡ ΖΘ πρὸς ΖΔ καὶ ἡ ΖΔ πρὸς Ε. κείσθω τῇ μὲν ΘΖ ἴση ἡ Β, τῇ δὲ ΔΖ ἴση ἡ Γ. ἔστιν ἄρα, ὡς ἡ Α πρὸς τὴν Β, ἡ Β πρὸς τὴν Γ καὶ ἡ Γ πρὸς Ε. αἱ Α, Β, Γ, Ε ἄρα ἑξῆς ἀνάλογόν εἰσιν· ὅπερ ἔδει εὑρεῖν.

there be drawn through Δ a parabola whose axis is ΔΗ, and *latus rectum* Α, and let the squares of the ordinates drawn at right angles to ΔΗ be equal to the areas applied to Α having as their sides the straight lines cut off by them towards Δ. Let it be drawn, and let it be ΔΘ, and let ΔΚ be perpendicular [to ΔΗ], and in the asymptotes ΚΔ, ΔΖ let there be drawn a hyperbola, such that the straight lines drawn parallel to ΚΔ, ΔΖ will make an area equal to the rectangle comprehended by Α, Ε. It will then cut the parabola. Let it cut at Θ, and let ΘΚ, ΘΖ be drawn perpendicular. Since then

$$Z\Theta^2 = A \cdot \Delta Z,$$

it follows that

$$A : Z\Theta = \Theta Z : Z\Delta.$$

Again, since $A \cdot E = \Theta Z \cdot Z\Delta,$

it follows that

$$A : Z\Theta = Z\Delta : E.$$

But $A : Z\Theta = Z\Theta : Z\Delta.$

Therefore $A : Z\Theta = Z\Theta : Z\Delta = Z\Delta : E.$

Let Β be placed equal to ΘΖ, and Γ equal to ΔΖ. It follows that

$$A : B = B : \Gamma = \Gamma : E.$$

Α, Β, Γ, Ε are therefore in continuous proportion; which was to be found.[a]

[a] If a, x, y, b are in continuous proportion,

$$\frac{a}{x} = \frac{x}{y} = \frac{y}{b}, \text{ and } x^2 = ay,\ y^2 = bx,\ xy = ab.$$

Therefore x, y may be determined as the intersection of the parabola $y^2 = bx$ and the hyperbola $xy = ab$. This is the

Ibid. 84. 12–88. 2

Ἡ Ἀρχύτου εὕρησις, ὡς Εὔδημος ἱστορεῖ

Ἔστωσαν αἱ δοθεῖσαι δύο εὐθεῖαι αἱ ΑΔ, Γ· δεῖ δὴ τῶν ΑΔ, Γ δύο μέσας ἀνάλογον εὑρεῖν.

Γεγράφθω περὶ τὴν μείζονα τὴν ΑΔ κύκλος ὁ ΑΒΔΖ, καὶ τῇ Γ ἴση ἐνηρμόσθω ἡ ΑΒ καὶ ἐκβληθεῖσα συμπιπτέτω τῇ ἀπὸ τοῦ Δ ἐφαπτομένῃ τοῦ κύκλου κατὰ τὸ Π, παρὰ δὲ τὴν ΠΔΟ ἤχθω ἡ

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analytical expression of the solution given above, where $E=a$ and $A=b$. The proof is followed in the text of Eutocius by another solution, introduced with the word Ἄλλως, "Otherwise," determining x, y as the intersection of the parabolas $x^2=ay$, $y^2=bx$. See above, 270 n *a*.

This is the earliest known use of conic sections in the history of Greek mathematics, and Menaechmus is accordingly credited with their discovery. But the names parabola and hyperbola were not used by him; they are due to Apollonius; Menaechmus would have called them, with Archimedes, sections of a right-angled and obtuse-angled cone.

From the equations given above it follows that

$$x^2+y^2-bx-ay=0$$

is a circle passing through the points common to the parabolas

$$x^2=ay,\ y^2=bx.$$

It follows that x, y may be determined by the intersection of this circle with the hyperbola $xy=ab$.

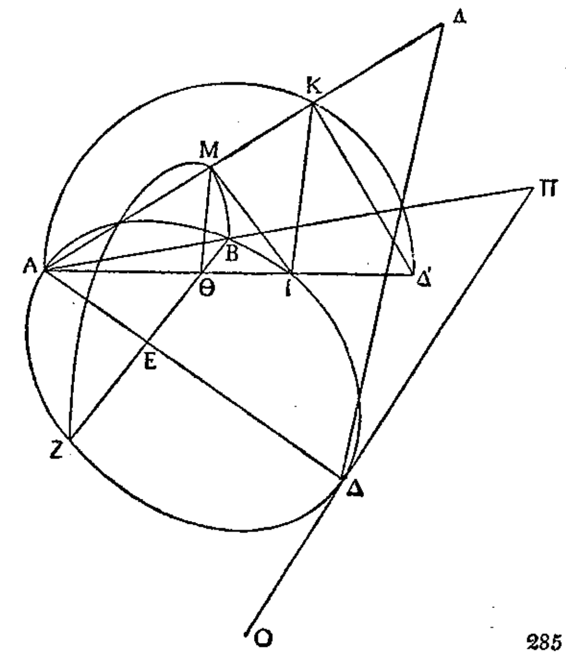
This is, in effect, the proof given by Heron, Philon and Apollonius. For, in the figure on p. 269, if ΔZ, ΔE are the co-ordinate axes, $AB=a$, $B\Gamma=b$, then $x^2+y^2-bx-ay=0$ is the circle passing through A, B, Γ, and $xy=ab$ is the hyperbola having ΔZ, ΔE as asymptotes and passing through B.

Ibid. 84. 12–88. 2

(v.) *The Solution of Archytas, according to Eudemus*

Let the two given straight lines be ΑΔ, Γ; it is required to find two mean proportionals between ΑΔ, Γ.

Let the circle ΑΒΔΖ be described about the greater straight line ΑΔ, and let ΑΒ be inserted equal to Γ and let it be produced so as to meet at Π the tangent to the circle at Δ. Let ΒΕΖ be drawn parallel to ΠΔΟ,



ΒΕΖ, καὶ νενοήσθω ἡμικυλίνδριον ὀρθὸν ἐπὶ τοῦ ΑΒΔ ἡμικυκλίου, ἐπὶ δὲ τῆς ΑΔ ἡμικύκλιον ὀρθὸν ἐν τῷ τοῦ ἡμικυλινδρίου παραλληλογράμμῳ κείμενον· τοῦτο δὴ τὸ ἡμικύκλιον περιαγόμενον ὡς ἀπὸ τοῦ Δ ἐπὶ τὸ Β μένοντος τοῦ Α πέρατος τῆς διαμέτρου τεμεῖ τὴν κυλινδρικὴν ἐπιφάνειαν ἐν τῇ περιαγωγῇ καὶ γράψει ἐν αὐτῇ γραμμήν τινα. πάλιν δέ, ἐὰν τῆς ΑΔ μενούσης τὸ ΑΠΔ τρίγωνον περιενεχθῇ τὴν ἐναντίαν τῷ ἡμικυκλίῳ κίνησιν, κωνικὴν ποιήσει ἐπιφάνειαν τῇ ΑΠ εὐθείᾳ, ἣ δὴ περιαγομένη συμβαλεῖ τῇ κυλινδρικῇ γραμμῇ κατά τι σημεῖον· ἅμα δὲ καὶ τὸ Β περιγράψει ἡμικύκλιον ἐν τῇ τοῦ κώνου ἐπιφανείᾳ. ἐχέτω δὴ θέσιν κατὰ τὸν τόπον τῆς συμπτώσεως τῶν γραμμῶν τὸ μὲν κινούμενον ἡμικύκλιον ὡς τὴν τοῦ ΔΚΑ, τὸ δὲ ἀντιπεριαγόμενον τρίγωνον τὴν τοῦ ΔΛΑ, τὸ δὲ τῆς εἰρημένης συμπτώσεως σημεῖον ἔστω τὸ Κ, ἔστω δὲ καὶ διὰ τοῦ Β γραφόμενον ἡμικύκλιον τὸ ΒΜΖ, κοινὴ δὲ αὐτοῦ τομὴ καὶ τοῦ ΒΔΖΑ κύκλου ἔστω ἡ ΒΖ, καὶ ἀπὸ τοῦ Κ ἐπὶ τὸ τοῦ ΒΔΑ ἡμικυκλίου ἐπίπεδον κάθετος ἤχθω· πεσεῖται δὴ ἐπὶ τὴν τοῦ κύκλου περιφέρειαν διὰ τὸ ὀρθὸν ἑστάναι τὸν κύλινδρον. πιπτέτω καὶ ἔστω ἡ ΚΙ, καὶ ἡ ἀπὸ τοῦ Ι ἐπὶ τὸ Α ἐπιζευχθεῖσα συμβαλέτω τῇ ΒΖ κατὰ τὸ Θ, ἡ δὲ ΑΛ τῷ ΒΜΖ ἡμικυκλίῳ κατὰ τὸ Μ, ἐπεζεύχθωσαν δὲ καὶ αἱ ΚΔ, ΜΙ, ΜΘ. ἐπεὶ οὖν ἑκάτερον τῶν ΔΚΑ, ΒΜΖ ἡμικυκλίων ὀρθόν ἐστι πρὸς τὸ ὑποκείμενον ἐπίπεδον, καὶ ἡ κοινὴ ἄρα αὐτῶν τομὴ ἡ ΜΘ πρὸς ὀρθάς ἐστι τῷ τοῦ κύκλου ἐπιπέδῳ· ὥστε καὶ πρὸς τὴν ΒΖ ὀρθή ἐστιν ἡ ΜΘ. τὸ ἄρα ὑπὸ τῶν ΒΘΖ, του-

and let a right half-cylinder be conceived upon the semicircle ABΔ, and on AΔ a right semicircle lying in the parallelogram of the half-cylinder. When this semicircle is moved about from Δ to B, the end point A of the diameter remaining fixed, it will cut the cylindrical surface in its motion and will describe in it a certain curve. Again, if AΔ be kept stationary and the triangle AΠΔ be moved about with an opposite motion to that of the semicircle, it will make a conic surface by means of the straight line AΠ, which in its motion will meet the curve on the cylinder in a certain point; at the same time B will describe a semicircle on the surface of the cone. Corresponding to the point in which the curves meet let the moving semicircle take up a position Δ′KA,[a] and the triangle moved in the opposite direction a position ΔΛA; let the point of the aforesaid meeting be K, and let BMZ be the semicircle described through B, and let BZ be the section common to it and the circle BΔZA, and let there be drawn from K a perpendicular upon the plane of the semicircle BΔA; it will fall upon the circumference of the circle because the cylinder is right. Let it fall, and let it be KI, and let the straight line joining I to A meet BZ in Θ; let AΛ meet the semicircle BMZ in M, and let KΔ, MI, MΘ be joined. Therefore since each of the semicircles Δ′KA, BMZ is at right angles to the underlying plane, their common section MΘ is also at right angles to the plane of the circle; so that MΘ is also at right angles to BZ. Therefore the rectangle contained by

[a] In the text and figure of the MSS. the same letter is used to indicate the initial and final positions of Δ; for convenience they are distinguished in the figure and translation as Δ, Δ′. It would make the figure easier to grasp if Λ could be written Π′ (for Λ is the final position of Π).

τέστι τὸ ὑπὸ ΑΘΙ, ἴσον ἐστὶ τῷ ἀπὸ ΜΘ· ὅμοιον ἄρα ἐστὶ τὸ ΑΜΙ τρίγωνον ἑκατέρῳ τῶν ΜΙΘ, ΜΑΘ, καὶ ὀρθὴ ἡ ὑπὸ ΙΜΑ. ἔστιν δὲ καὶ ἡ ὑπὸ ΔΚΑ ὀρθή. παράλληλοι ἄρα εἰσὶν αἱ ΚΔ, ΜΙ, καὶ ἔσται ἀνάλογον, ὡς ἡ ΔΑ πρὸς ΑΚ, τουτέστιν ἡ ΚΑ πρὸς ΑΙ, οὕτως ἡ ΙΑ πρὸς ΑΜ, διὰ τὴν ὁμοιότητα τῶν τριγώνων. τέσσαρες ἄρα αἱ ΔΑ, ΑΚ, ΑΙ, ΑΜ ἑξῆς ἀνάλογόν εἰσιν. καί ἐστιν ἡ ΑΜ ἴση τῇ Γ, ἐπεὶ καὶ τῇ ΑΒ· δύο ἄρα δοθεισῶν τῶν ΑΔ, Γ δύο μέσαι ἀνάλογον ηὕρηνται αἱ ΑΚ, ΑΙ.

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[a] The above solution is a remarkable achievement when it is remembered that Archytas flourished in the first half of the fourth century B.C., at which time Greek geometry was still in its infancy. It is quite easy, however, for us to represent the solution analytically. If AΔ is taken as the axis of x, the perpendicular to AΔ at A in the plane of the paper as the axis of y, and the perpendicular to these lines as the axis of z, and if $A\Delta = a$, $\Gamma = b$, then the point K is determined as the intersection of the following three curves:

(1) The cylinder $x^2 + y^2 = ax$,

(2) the curve formed by the motion of the half-circle about A (a tore of inner diameter nil)

$$x^2 + y^2 + z^2 = a\sqrt{x^2 + y^2},$$

(3) the cone $x^2 + y^2 + z^2 = \frac{a^2}{b^2}x^2$.

BΘ, ΘZ, which is the same as the rectangle contained by AΘ, ΘI, is equal to the square on MΘ; therefore the triangle AMI is similar to each of the triangles MIΘ, MAΘ, and the angle IMA is right. The angle Δ′KA is also right. Therefore KΔ′, MI are parallel, and owing to the similarity of the triangles the following proportion holds:

$$\Delta'A : AK = KA : AI = IA : AM.$$

Therefore the four straight lines ΔA, AK, AI, AM are in continuous proportion. And AM is equal to Γ, since it is equal to AB; therefore to the two given straight lines AΔ, Γ, two mean proportionals, AK, AI, have been found.[a]

Since K is the point of intersection,

$$AK = \sqrt{x^2 + y^2 + z^2},\ AI = \sqrt{x^2 + y^2}.$$

From (2) it follows directly that

$$AK^2 = a \cdot AI$$

i.e.,

$$\frac{a}{AK} = \frac{AK}{AI}.$$

From (1) and (3) it follows that

$$x^2 + y^2 + z^2 = \frac{(x^2 + y^2)^2}{b^2}$$

$$\therefore \quad \sqrt{x^2 + y^2 + z^2} = \frac{x^2 + y^2}{b}$$

i.e.,

$$AK = \frac{AI^2}{b}$$

or

$$\frac{AK}{AI} = \frac{AI}{b}$$

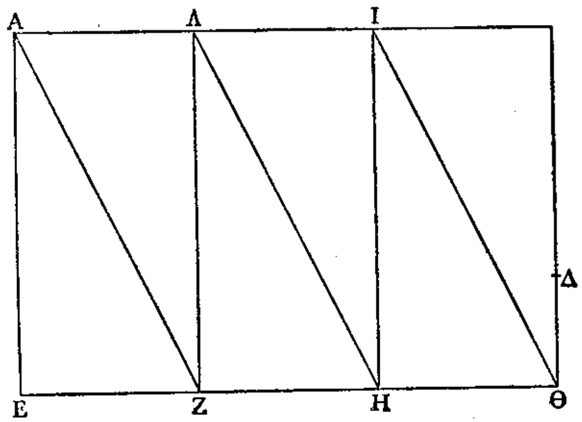
$$\therefore \quad \frac{a}{AK} = \frac{AK}{AI} = \frac{AI}{b},$$

and AK, AI are mean proportionals between a and b.

Ibid. 88. 3–96. 27

Ὡς Ἐρατοσθένης . . .

Δεδόσθωσαν δύο ἄνισοι εὐθεῖαι, ὧν δεῖ δύο μέσας ἀνάλογον εὑρεῖν ἐν συνεχεῖ ἀναλογίᾳ, αἱ ΑΕ, ΔΘ, καὶ κείσθω ἐπί τινος εὐθείας τῆς ΕΘ



πρὸς ὀρθὰς ἡ ΑΕ, καὶ ἐπὶ τῆς ΕΘ τρία συνεστάτω παραλληλόγραμμα ἐφεξῆς τὰ ΑΖ, ΖΙ, ΙΘ, καὶ ἤχθωσαν διάμετροι ἐν αὐτοῖς αἱ ΑΖ, ΛΗ, ΙΘ· ἔσονται δὴ αὗται παράλληλοι. μένοντος δὴ τοῦ μέσου παραλληλογράμμου τοῦ ΖΙ συνωσθήτω τὸ μὲν ΑΖ ἐπάνω τοῦ μέσου, τὸ δὲ ΙΘ ὑποκάτω, καθάπερ ἐπὶ τοῦ δευτέρου σχήματος, ἕως οὗ γένηται τὰ Α, Β, Γ, Δ κατ' εὐθεῖαν, καὶ διήχθω διὰ τῶν Α, Β, Γ, Δ σημείων εὐθεῖα καὶ συμπιπτέτω τῇ ΕΘ ἐκβληθείσῃ κατὰ τὸ Κ· ἔσται δή, ὡς ἡ ΑΚ πρὸς ΚΒ, ἐν μὲν ταῖς ΑΕ, ΖΒ παραλ-

Ibid. 88. 3–96. 27 [a]

(vi.) *The Solution of Eratosthenes* . . .

Let there be given two unequal straight lines ΑΕ, ΔΘ between which it is required to find two mean proportionals in continued proportion, and let ΑΕ be placed at right angles to the straight line ΕΘ, and upon ΕΘ let there be erected three successive parallelograms [b] ΑΖ, ΖΙ, ΙΘ, and let the diagonals ΑΖ, ΛΗ, ΙΘ be drawn therein ; these will be parallel. While the middle parallelogram ΖΙ remains stationary, let the other two approach each other, ΑΖ above the middle one, ΙΘ below it, as in the second figure,[c] until Α, Β, Γ, Δ lie along a straight line, and let a straight line be drawn through the points Α, Β, Γ, Δ, and let it meet ΕΘ produced in Κ ; it will follow that in the parallels ΑΕ, ΖΒ

$$AK : KB = EK : KZ$$

[a] This is the letter falsely purporting to be by Eratosthenes of which the beginning has already been cited, *supra*, pp. 256-261. The extract here given (δεδόσθωσαν . . .) starts in Heiberg's text at 90. 30. Eratosthenes' solution is given, with variations, by Pappus, *Collection* iii. 7, ed. Hultsch 56. 18–58. 22.

[b] Pappus says triangles in his account ; it makes no difference.

[c] See p. 294.

λήλοις ἡ ΕΚ πρὸς ΚΖ, ἐν δὲ ταῖς ΑΖ, ΒΗ παραλλήλοις ἡ ΖΚ πρὸς ΚΗ. ὡς ἄρα ἡ ΑΚ πρὸς ΚΒ, ἡ ΕΚ πρὸς ΚΖ καὶ ἡ ΚΖ πρὸς ΚΗ. πάλιν, ἐπεί ἐστιν, ὡς ἡ ΒΚ πρὸς ΚΓ, ἐν μὲν ταῖς ΒΖ, ΓΗ παραλλήλοις ἡ ΖΚ πρὸς ΚΗ, ἐν δὲ ταῖς ΒΗ, ΓΘ παραλλήλοις ἡ ΗΚ πρὸς ΚΘ, ὡς ἄρα ἡ ΒΚ πρὸς ΚΓ, ἡ ΖΚ πρὸς ΚΗ καὶ ἡ ΗΚ πρὸς ΚΘ. ἀλλ' ὡς ἡ ΖΚ πρὸς ΚΗ, ἡ ΕΚ πρὸς ΚΖ· καὶ ὡς ἄρα ἡ ΕΚ πρὸς ΚΖ, ἡ ΖΚ πρὸς ΚΗ καὶ ἡ ΗΚ πρὸς ΚΘ. ἀλλ' ὡς ἡ ΕΚ πρὸς ΚΖ, ἡ ΑΕ πρὸς ΒΖ, ὡς δὲ ἡ ΖΚ πρὸς ΚΗ, ἡ ΒΖ πρὸς ΓΗ, ὡς δὲ ἡ ΗΚ πρὸς ΚΘ, ἡ ΓΗ πρὸς ΔΘ· καὶ ὡς ἄρα ἡ ΑΕ πρὸς ΒΖ, ἡ ΒΖ πρὸς ΓΗ καὶ ἡ ΓΗ πρὸς ΔΘ. ηὕρηνται ἄρα τῶν ΑΕ, ΔΘ δύο μέσαι ἥ τε ΒΖ καὶ ἡ ΓΗ.

Ταῦτα οὖν ἐπὶ τῶν γεωμετρουμένων ἐπιφανειῶν ἀποδέδεικται· ἵνα δὲ καὶ ὀργανικῶς δυνώμεθα τὰς δύο μέσας λαμβάνειν, διαπήγνυται πλινθίον ξύλινον ἢ ἐλεφάντινον ἢ χαλκοῦν ἔχον τρεῖς πινακίσκους ἴσους ὡς λεπτοτάτους, ὧν ὁ μὲν μέσος ἐνήρμοσται, οἱ δὲ δύο ἐπωστοί εἰσιν ἐν χολέδραις, τοῖς δὲ μεγέθεσιν καὶ ταῖς συμμετρίαις ὡς ἕκαστοι ἑαυτοὺς πείθουσιν· τὰ μὲν γὰρ τῆς ἀποδείξεως ὡσαύτως συντελεῖται· πρὸς δὲ τὸ ἀκριβέστερον λαμβάνεσθαι τὰς γραμμὰς φιλοτεχνητέον, ἵνα ἐν τῷ συνάγεσθαι τοὺς πινακίσκους παράλληλα διαμένῃ πάντα καὶ ἄσχαστα καὶ ὁμαλῶς συναπτόμενα ἀλλήλοις.

Ἐν δὲ τῷ ἀναθήματι τὸ μὲν ὀργανικὸν χαλκοῦν ἐστιν καὶ καθήρμοσται ὑπ' αὐτὴν τὴν στεφάνην τῆς στήλης προσμεμολυβδοχοημένον, ὑπ' αὐτοῦ δὲ ἡ ἀπόδειξις συντομώτερον φραζομένη καὶ τὸ σχῆμα,

and in the parallels AZ, BH

$$AK : KB = ZK : KH.$$

Therefore $AK : KB = EK : KZ = KZ : KH$.

Again, since in the parallels BZ, ΓH

$$BK : K\Gamma = ZK : KH$$

and in the parallels BH, ΓΘ

$$BK : K\Gamma = HK : K\Theta,$$

therefore $BK : K\Gamma = ZK : KH = HK : K\Theta$.

But $ZK : KH = EK : KZ$, and therefore

$$EK : KZ = ZK : KH = HK : K\Theta.$$

But $EK : KZ = AE : BZ$, $ZK : KH = BZ : \Gamma H$, $HK : K\Theta = \Gamma H : \Delta\Theta$.

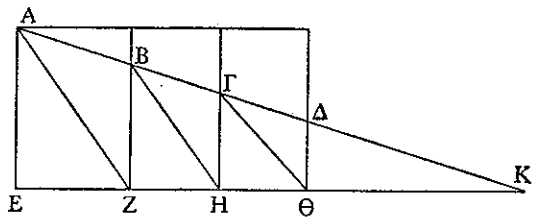
Therefore $AE : BZ = BZ : \Gamma H = \Gamma H : \Delta\Theta$.

Therefore between AE, ΔΘ two means, BZ, ΓH, have been found.

Such is the demonstration on geometrical surfaces; and in order that we may find the two means mechanically, a board of wood or ivory or bronze is pierced through, having on it three equal tablets, as smooth as possible, of which the midmost is fixed and the two outside run in grooves, their sizes and proportions being a matter of individual choice—for the proof is accomplished in the same manner; in order that the lines may be found with the greatest accuracy, the instrument must be skilfully made, so that when the tablets are moved everything remains parallel, smoothly fitting without a gap.

In the votive gift the instrument is of bronze and is fastened on with lead close under the crown of the pillar, and beneath it is a shortened form of the proof μετ' αὐτὸ δὲ ἐπίγραμμα. ὑπογεγράφθω οὖν σοι καὶ ταῦτα, ἵνα ἔχῃς καὶ ὡς ἐν τῷ ἀναθήματι. τῶν δὲ δύο σχημάτων τὸ δεύτερον γέγραπται ἐν τῇ στήλῃ.

"Δύο τῶν δοθεισῶν εὐθειῶν δύο μέσας ἀνάλογον εὑρεῖν ἐν συνεχεῖ ἀναλογίᾳ. δεδόσθωσαν αἱ ΑΕ, ΔΘ. συνάγω δὴ τοὺς ἐν τῷ ὀργάνῳ πίνακας, ἕως ἂν κατ' εὐθεῖαν γένηται τὰ Α, Β, Γ, Δ σημεῖα. νοείσθω δή, ὡς ἔχει ἐπὶ τοῦ δευτέρου σχήματος. ἔστιν ἄρα, ὡς ἡ ΑΚ πρὸς ΚΒ, ἐν μὲν ταῖς ΑΕ, ΒΖ παραλλήλοις ἡ ΕΚ πρὸς ΚΖ, ἐν δὲ ταῖς ΑΖ, ΒΗ ἡ ΖΚ πρὸς ΚΗ· ὡς ἄρα ἡ ΕΚ πρὸς ΚΖ, ἡ



ΚΖ πρὸς ΚΗ. ὡς δὲ αὗται πρὸς ἀλλήλας, ἥ τε ΑΕ πρὸς ΒΖ καὶ ἡ ΒΖ πρὸς ΓΗ. ὡσαύτως δὲ δείξομεν, ὅτι καί, ὡς ἡ ΖΒ πρὸς ΓΗ, ἡ ΓΗ πρὸς ΔΘ· ἀνάλογον ἄρα αἱ ΑΕ, ΒΖ, ΓΗ, ΔΘ. ηὕρηνται ἄρα δύο τῶν δοθεισῶν δύο μέσαι.

"Ἐὰν δὲ αἱ δοθεῖσαι μὴ ἴσαι ὦσιν ταῖς ΑΕ, ΔΘ, ποιήσαντες αὐταῖς ἀνάλογον τὰς ΑΕ, ΔΘ τούτων ληψόμεθα τὰς μέσας καὶ ἐπανοίσομεν ἐπ' ἐκείνας, καὶ ἐσόμεθα πεποιηκότες τὸ ἐπιταχθέν. ἐὰν δὲ πλείους μέσας ἐπιταχθῇ εὑρεῖν, ἀεὶ ἑνὶ πλείους πινακίσκους καταστησόμεθα ἐν τῷ ὀργανίῳ τῶν ληφθησομένων μέσων· ἡ δὲ ἀπόδειξις ἡ αὐτή·

and the figure, and along with this is an epigram. These also shall be written below for you, in order that you may have what is on the votive gift. Of the two figures, the second is that which is inscribed on the pillar.[a]

"Between two given straight lines to find two means in continuous proportion. Let ΑΕ, ΔΘ be the given straight lines. Then I move the tables in the instrument until the points Α, Β, Γ, Δ are in the same straight line. Let this be pictured as in the second figure. Then ΑΚ : ΚΒ is equal, in the parallels ΑΕ, ΒΖ, to ΕΚ : ΚΖ, and in the parallels ΑΖ, ΒΗ to ΖΚ : ΚΗ; therefore ΕΚ : ΚΖ = ΚΖ : ΚΗ. Now this is also the ratio ΑΕ : ΒΖ and ΒΖ : ΓΗ. Similarly we shall show that ΖΒ : ΓΗ = ΓΗ : ΔΘ; ΑΕ, ΒΖ, ΓΗ, ΔΘ are therefore proportional. Between the two given straight lines two means have therefore been found.

"If the given straight lines are not equal to ΑΕ, ΔΘ, by making ΑΕ, ΔΘ proportional to them and taking the means between these and then going back to the original lines, we shall do what was enjoined. If it is required to find more means, we shall continually insert more tables in the instrument according to the number of means to be taken; and the proof is the same.

[a] The short proof and epigram which follow are presumably the genuine work of Eratosthenes, being taken from the votive gift. The reference to the *second* figure cannot, however, be genuine as there was only one figure on the votive offering; perhaps δεύτερον should be omitted.

"Εἰ κύβον ἐξ ὀλίγου διπλήσιον, ὦγαθε, τεύχειν
  φράζεαι ἢ στερεὴν πᾶσαν ἐς ἄλλο φύσιν
εὖ μεταμορφῶσαι, τόδε τοι πάρα, κἂν σύ γε μάνδρην
  ἢ σιρὸν ἢ κοίλου φρείατος εὐρὺ κύτος
τῇδ' ἀναμετρήσαιο, μέσας ὅτε τέρμασιν ἄκροις
  συνδρομάδας δισσῶν ἐντὸς ἕλῃς κανόνων.
μηδὲ σύ γ' Ἀρχύτεω δυσμήχανα ἔργα κυλίνδρων
  μηδὲ Μεναιχμείους κωνοτομεῖν τριάδας
διζήσῃ, μηδ' εἴ τι θεουδέος Εὐδόξοιο
  καμπύλον ἐγ γραμμαῖς εἶδος ἀναγράφεται.
τοῖσδε γὰρ ἐν πινάκεσσι μεσόγραφα μυρία τεύχοις
  ῥεῖά κεν ἐκ παύρου πυθμένος ἀρχόμενος.
εὐαίων, Πτολεμαῖε, πατὴρ ὅτι παιδὶ συνηβῶν
  πάνθ', ὅσα καὶ Μούσαις καὶ βασιλεῦσι φίλα,
αὐτὸς ἐδωρήσω· τὸ δ' ἐς ὕστερον, οὐράνιε Ζεῦ,
  καὶ σκήπτρων ἐκ σῆς ἀντιάσειε χερός.
καὶ τὰ μὲν ὡς τελέοιτο, λέγοι δέ τις ἄνθεμα λεύσσων
  τοῦ Κυρηναίου τοῦτ' Ἐρατοσθένεος."

Ibid. 98. 1-7

Ὡς Νικομήδης ἐν τῷ Περὶ κογχοειδῶν γραμμῶν

Γράφει δὲ καὶ Νικομήδης ἐν τῷ ἐπιγεγραμμένῳ πρὸς αὐτοῦ Περὶ κογχοειδῶν συγγράμματι ὀργάνου κατασκευὴν τὴν αὐτὴν ἀποπληροῦντος χρείαν, ἐφ' ᾧ καὶ μεγάλα μὲν σεμνυνόμενος φαίνεται ὁ ἀνήρ, πολλὰ δὲ τοῖς Ἐρατοσθένους ἐπεγγελῶν εὑρήμασιν

[a] Or " with a small effort," Heiberg.

[b] Perhaps so called because there are three conic sections —of an acute-angled, right-angled and obtuse-angled cone

" If, good friend, thou thinkest to produce from a small [cube] [a] one double thereof, or duly to change any solid figure into another nature, this is in thy power, and thou canst measure a byre or corn-pit or the broad basin of a hollow well by this method, when thou takest between two rulers means converging with their extreme ends. Do not seek to do the difficult business of the cylinders of Archytas, or to cut the cone in the triads [b] of Menaechmus, or to produce any such curved form in lines as is described by the divine Eudoxus. Indeed, on these tablets thou couldst easily find a thousand means, beginning from a small base. Happy art thou, O Ptolemy, a father who lives his son's life in all things, in that thou hast given him such things as are dear to the Muses and kings; and in the future, O heavenly Zeus, may he also receive the sceptre from thy hands. May this prayer be fulfilled, and may anyone seeing this votive offering say: This is the gift of Eratosthenes of Cyrene."

Ibid. 98. 1-7

(vii.) *The Solution of Nicomedes in his Book " On Conchoidal Lines "* [c]

Nicomedes also describes, in the book written by him *On Conchoids*, the construction of an instrument fulfilling the same purpose, upon which it appears he prided himself exceedingly, greatly deriding the

(ellipse, parabola and hyperbola). If so, this proves that Menaechmus discovered the ellipse as well as the other two.

[c] It follows from this extract that Nicomedes was later than Eratosthenes; and as Apollonius called a certain curve " sister of the *cochloid* " (*infra*, p. 334), he must have been younger than Apollonius. He was therefore born about 270 B.C.

ὡς ἀμηχάνοις τε ἅμα καὶ γεωμετρικῆς ἕξεως ἐστερημένοις.

Papp. *Coll.* iv. 26. 39–28. 43, ed. Hultsch 242. 13–250. 25

κϛʹ. Εἰς τὸν διπλασιασμὸν τοῦ κύβου παράγεταί τις ὑπὸ Νικομήδους γραμμὴ καὶ γένεσιν ἔχει τοιαύτην.

Ἐκκείσθω εὐθεῖα ἡ ΑΒ, καὶ αὐτῇ πρὸς ὀρθὰς ἡ ΓΔΖ, καὶ εἰλήφθω τι σημεῖον ἐπὶ τῆς ΓΔΖ δοθὲν τὸ Ε, καὶ μένοντος τοῦ Ε σημείου ἐν ᾧ ἐστιν τόπῳ ἡ ΓΔΕΖ εὐθεῖα φερέσθω κατὰ τῆς ΑΔΒ εὐθείας ἑλκομένη διὰ τοῦ Ε σημείου οὕτως ὥστε διὰ παντὸς φέρεσθαι τὸ Δ ἐπὶ τῆς ΑΒ εὐθείας καὶ μὴ ἐκπίπτειν ἑλκομένης τῆς ΓΔΕΖ διὰ τοῦ Ε. τοιαύτης δὴ κινήσεως γενομένης ἐφ' ἑκάτερα φανερὸν ὅτι τὸ Γ σημεῖον γράφει γραμμὴν οἵα ἐστὶν ἡ ΛΓΜ, καὶ ἔστιν αὐτῆς τὸ σύμπτωμα τοιοῦτον. ὡς ἂν εὐθεῖα προσπίπτῃ τις ἀπὸ τοῦ Ε σημείου πρὸς τὴν γραμμήν, τὴν ἀπολαμβανομένην μεταξὺ τῆς τε ΑΒ εὐθείας καὶ τῆς ΛΓΜ γραμμῆς ἴσην εἶναι τῇ ΓΔ εὐθείᾳ· μενούσης γὰρ τῆς ΑΒ καὶ μένοντος τοῦ Ε σημείου, ὅταν γένηται τὸ Δ ἐπὶ τὸ Η, ἡ ΓΔ εὐθεῖα τῇ ΗΘ ἐφαρμόσει καὶ τὸ Γ σημεῖον ἐπὶ τὸ Θ ⟨πεσεῖται⟩[1]· ἴση ἄρα ἐστὶν ἡ ΓΔ τῇ ΗΘ. ὁμοίως καὶ ἐὰν ἑτέρα τις

[1] πεσεῖται add. Hultsch.

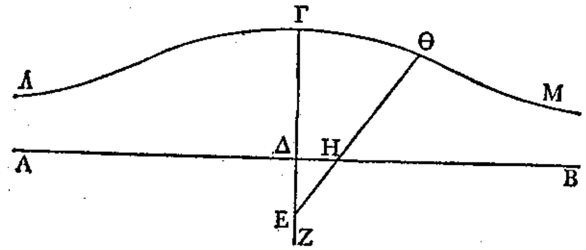
[a] Eutocius proceeds to describe Nicomedes' solution; we shall give an alternative account by Pappus.

discoveries of Eratosthenes as impracticable and lacking in geometrical sense.[a]

Pappus, *Collection* iv. 26. 39–28. 43, ed. Hultsch 242. 13–250. 25

26. For the duplication of the cube a certain line is drawn by Nicomedes and generated in this way.

Let there be a straight line AB, with ΓΔZ at right angles to it, and on ΓΔZ let there be taken a certain



given point E, and while the point E remains in the same position let the straight line ΓΔEZ be drawn through the point E and moved about the straight line AΔB in such a way that Δ always moves along the straight line AB and does not fall beyond it while ΓΔEZ is drawn through E. The motion being after this fashion on either side, it is clear that the point Γ will describe a curve such as ΛΓM, and its property is of this nature: when any straight line drawn from the point E falls upon the curve, the portion cut off between the straight line AB and the curve ΛΓM is equal to the straight line ΓΔ; for AB is stationary and the point E fixed, and when Δ goes to H, the straight line ΓΔ will coincide with HΘ and the point Γ will fall upon Θ: therefore ΓΔ is equal to HΘ.

ἀπὸ τοῦ Ε σημείου πρὸς τὴν γραμμὴν προσπέσῃ, τὴν ἀποτεμνομένην ὑπὸ τῆς γραμμῆς καὶ τῆς ΑΒ εὐθείας ἴσην ποιήσει τῇ ΓΔ [ἐπειδὴ ταύτῃ ἴσαι εἰσὶν αἱ προσπίπτουσαι].[1] καλείσθω δέ, φησίν, ἡ μὲν ΑΒ εὐθεῖα κανών, τὸ δὲ σημεῖον πόλος, διάστημα δὲ ἡ ΓΔ, ἐπειδὴ ταύτῃ ἴσαι εἰσὶν αἱ προσπίπτουσαι πρὸς τὴν ΛΓΜ γραμμήν, αὐτὴ δὲ ἡ ΛΓΜ γραμμὴ κοχλοειδὴς πρώτη (ἐπειδὴ καὶ ἡ δευτέρα καὶ ἡ τρίτη καὶ ἡ τετάρτη ἐκτίθεται εἰς ἄλλα θεωρήματα χρησιμεύουσαι).

κζʹ. Ὅτι δὲ ὀργανικῶς δύναται γράφεσθαι ἡ γραμμὴ καὶ ἐπ' ἔλαττον ἀεὶ συμπορεύεται τῷ κανόνι, τουτέστιν ὅτι πασῶν τῶν ἀπό τινων σημείων τῆς ΛΓΘ γραμμῆς ἐπὶ τὴν ΑΒ εὐθεῖαν καθέτων μεγίστη ἐστὶν ἡ ΓΔ κάθετος, ἀεὶ δὲ ἡ ἔγγιον τῆς ΓΔ ἀγομένη κάθετος τῆς ἀπώτερον μείζων ἐστίν, καὶ ὅτι, εἰς τὸν μεταξὺ τόπον τοῦ κανόνος καὶ τῆς κοχλοειδοῦς ἐάν τις ᾖ εὐθεῖα, ἐκβαλλομένη τμηθήσεται ὑπὸ τῆς κοχλοειδοῦς, αὐτὸς ἀπέδειξεν ὁ Νικομήδης, καὶ ἡμεῖς ἐν τῷ εἰς τὸ Ἀνάλημμα Διοδώρου, τρίχα τεμεῖν τὴν γωνίαν βουλόμενοι, κεχρήμεθα τῇ προειρημένῃ γραμμῇ.

[1] ἐπειδὴ . . . προσπίπτουσαι "ex proximis inepte huc translata" del. Hultsch.

[a] Let a be the interval or constant intercept between the curve and the base, and b the distance from the pole to the base (ΕΔ). If Θ is any point on the curve, and $E\Theta = \tau$, $\angle \Gamma E\Theta = \phi$, then the fundamental equation of the curve is

$$\tau = b \sec \phi + a.$$

If a is measured *backwards* from the base towards the pole, then another conchoidal figure is obtained on the same side of the base as the pole, having for its fundamental equation

$$\tau = b \sec \phi - a.$$

This takes three forms according as a is greater than,

Similarly, if any other straight line drawn from the point E falls upon the curve, the portion cut off by the curve and the straight line AB will make a straight line equal to ΓΔ. Now, says he, let the straight line AB be called the ruler, the point [E] the pole, ΓΔ the interval, since the straight lines falling upon the line ΛΓΜ are equal to it, and let the curve ΛΓΜ itself be called the first cochloidal line (since there are second and third and fourth cochloids which are useful for other theorems).[a]

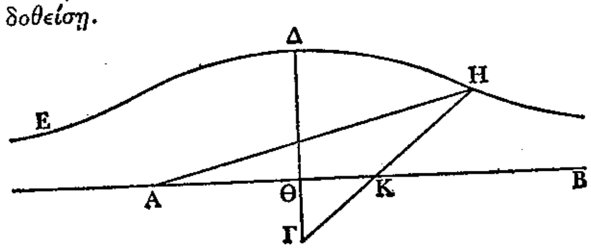
27. Nicomedes himself proved that the curve can be described mechanically, and that it continually approaches closer to the ruler—which is equivalent to saying that of all the perpendiculars drawn from points on the line ΛΓΘ to the straight line AB the greatest is the perpendicular ΓΔ, while the perpendicular drawn nearer to ΓΔ is always greater than the more remote; he also proved that any straight line in the space between the ruler and the cochloid will be cut, when produced, by the cochloid; and we used the aforesaid line in the commentary on the *Analemma*[b] of Diodorus when we sought to trisect an angle.

equal to, or less than b. These three forms are probably the "second, third and fourth cochloids," but we have no direct information. When a is greater than b, the curve has a loop at the pole; when a equals b, there is a cusp at the pole; when a is less than b, there is no double point.

The original name of the curve would appear to be the *cochloid* (κοχλοειδὴς γραμμή), as it is called by Pappus, from a supposed resemblance to a shell-fish (κόχλος). Later it was called the *conchoid* (κογχοειδὴς γραμμή), the "mussel-like" curve.

[b] Diodorus of Alexandria lived in the time of Caesar and is commemorated in the *Anthology* (xiv. 139) as a maker of gnomons. Ptolemy also wrote an *Analemma*, whose object is a graphic representation on a plane of parts of the heavenly sphere.

Διὰ δὴ τῶν εἰρημένων φανερὸν ὡς δυνατόν ἐστιν γωνίας δοθείσης ὡς τῆς ὑπὸ ΗΑΒ καὶ σημείου ἐκτὸς αὐτῆς τοῦ Γ διάγειν τὴν ΓΗ καὶ ποιεῖν τὴν ΚΗ μεταξὺ τῆς γραμμῆς καὶ τῆς ΑΒ ἴσην τῇ δοθείσῃ.



Ἤχθω κάθετος ἀπὸ τοῦ Γ σημείου ἐπὶ τὴν ΑΒ ἡ ΓΘ καὶ ἐκβεβλήσθω, καὶ τῇ δοθείσῃ ἴση ἔστω ἡ ΔΘ, καὶ πόλῳ μὲν τῷ Γ, διαστήματι δὲ τῷ δοθέντι, τουτέστιν τῇ ΔΘ, κανόνι δὲ τῷ ΑΒ γεγράφθω κοχλοειδὴς γραμμὴ πρώτη ἡ ΕΔΗ· συμβάλλει ἄρα τῇ ΑΗ διὰ τὸ προλεχθέν. συμβαλλέτω κατὰ τὸ Η, καὶ ἐπεζεύχθω ἡ ΓΗ· ἴση ἄρα καὶ ἡ ΚΗ τῇ δοθείσῃ.

κη′. Τινὲς δὲ τῆς χρήσεως ἕνεκα παρατιθέντες κανόνα τῷ Γ κινοῦσιν αὐτόν, ἕως ἂν ἐκ τῆς πείρας ἡ μεταξὺ ἀπολαμβανομένη τῆς ΑΒ εὐθείας καὶ τῆς ΕΔΗ γραμμῆς ἴση γένηται τῇ δοθείσῃ· τούτου γὰρ ὄντος τὸ προκείμενον ἐξ ἀρχῆς δείκνυται (λέγω δὲ κύβος κύβου διπλάσιος εὑρίσκεται). πρότερον δὲ δύο δοθεισῶν εὐθειῶν δύο μέσαι κατὰ τὸ συνεχὲς ἀνάλογον λαμβάνονται, ὧν ὁ μὲν Νικομήδης τὴν κατασκευὴν ἐξέθετο μόνον, ἡμεῖς

Now by what has been said it is clear that if there is an angle, such as HAB, and a point Γ outside the angle, it is possible so to draw ΓH as to make KH between the line and AB equal to a given straight line.

Let ΓΘ be drawn from the point Γ perpendicular to AB and produced to Δ so that ΔΘ is equal to the given straight line, and with Γ for pole, the given straight line, that is ΔΘ, for interval, and AB for ruler let the first cochloid EΔH be drawn; then by what has been said above it will meet AH; let it meet it in H, and let ΓH be joined; KH will therefore be equal to the given straight line.

28. Some people, following [a more convenient] usage, apply a ruler to Γ and move it until by trial the portion between the straight line AB and the line EΔH becomes equal to the given straight line; and when this is done the problem which was posed at the outset is solved (I mean a cube which is double of a cube is found). But first two means in continuous proportion are taken between two given straight lines; Nicomedes explained only the construction necessary

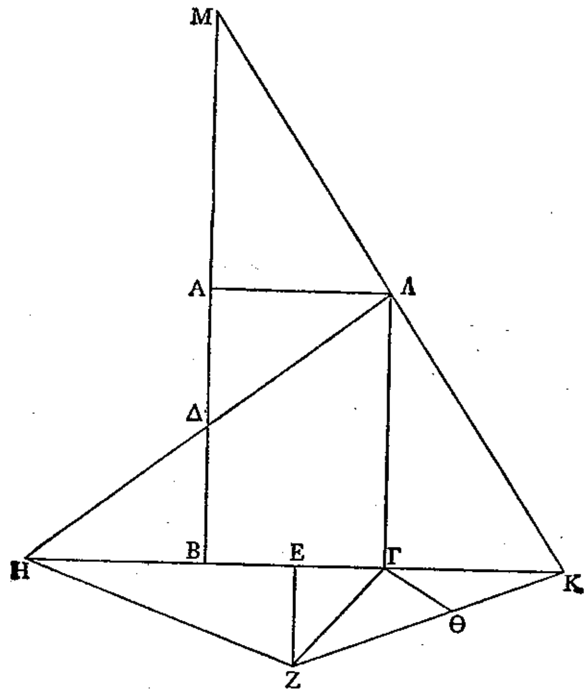
δὲ καὶ τὴν ἀπόδειξιν ἐφηρμόσαμεν τῇ κατασκευῇ τὸν τρόπον τοῦτον.

Δεδόσθωσαν γὰρ δύο εὐθεῖαι αἱ ΓΛ, ΛΑ πρὸς ὀρθὰς ἀλλήλαις, ὧν δεῖ δύο μέσας ἀνάλογον κατὰ τὸ συνεχὲς εὑρεῖν, καὶ συμπεπληρώσθω τὸ ΑΒΓΛ παραλληλόγραμμον, καὶ τετμήσθω δίχα ἑκατέρα τῶν ΑΒ, ΒΓ τοῖς Δ, Ε σημείοις, καὶ ἐπιζευχθεῖσα

[a] The proof is given by Eutocius with very few variations (pp. 104-106) and also in another place by Pappus himself (iii. 8, ed. Hultsch 58. 23–62. 13, with several differences). In iii. 8 the straight lines are called ΔΓ, ΔΑ, whereas here and in the passage from Eutocius the MSS. have ΓΛ, ΛΑ. Wherever we have Λ here, it is reasonably certain that Pappus wrote Δ, and *vice versa*.

for doing this, but we have supplied a proof to the construction in this manner.

Let [a] there be given two straight lines ΓΛ, ΛΑ at right angles to each other between which it is required



to find two means in continuous proportion, and let the parallelogram ΑΒΓΛ be completed, and let each of the straight lines ΑΒ, ΒΓ be bisected at the points μὲν ἡ ΔΛ ἐκβεβλήσθω καὶ συμπιπτέτω τῇ ΓΒ ἐκβληθείσῃ κατὰ τὸ Η, τῇ δὲ ΒΓ πρὸς ὀρθὰς ἡ ΕΖ, καὶ προσβεβλήσθω ἡ ΓΖ ἴση οὖσα τῇ ΑΔ, καὶ ἐπεζεύχθω ἡ ΖΗ καὶ αὐτῇ παράλληλος ἡ ΓΘ, καὶ γωνίας οὔσης τῆς ὑπὸ τῶν ΚΓΘ ἀπὸ δοθέντος τοῦ Ζ διήχθω ἡ ΖΘΚ ποιοῦσα ἴσην τὴν ΘΚ τῇ ΑΔ ἢ τῇ ΓΖ (τοῦτο γὰρ ὡς δυνατὸν ἐδείχθη διὰ τῆς κοχλοειδοῦς γραμμῆς), καὶ ἐπιζευχθεῖσα ἡ ΚΛ ἐκβεβλήσθω καὶ συμπιπτέτω τῇ ΑΒ ἐκβληθείσῃ κατὰ τὸ Μ· λέγω ὅτι ἐστὶν ὡς ἡ ΛΓ πρὸς ΚΓ, ἡ ΚΓ πρὸς ΜΑ, καὶ ἡ ΜΑ πρὸς τὴν ΑΛ.

Ἐπεὶ ἡ ΒΓ τέτμηται δίχα τῷ Ε καὶ πρόσκειται αὐτῇ ἡ ΚΓ, τὸ ἄρα ὑπὸ ΒΚΓ μετὰ τοῦ ἀπὸ ΓΕ ἴσον ἐστὶν τῷ ἀπὸ ΕΚ. κοινὸν προσκείσθω τὸ ἀπὸ ΕΖ· τὸ ἄρα ὑπὸ ΒΚΓ μετὰ τῶν ἀπὸ ΓΕΖ, τουτέστιν τοῦ ἀπὸ ΓΖ, ἴσον ἐστὶν τοῖς ἀπὸ ΚΕΖ, τουτέστιν τῷ ἀπὸ ΚΖ. καὶ ἐπεὶ ὡς ἡ ΜΑ πρὸς ΑΒ, ἡ ΜΛ πρὸς ΛΚ, ὡς δὲ ἡ ΜΛ πρὸς ΛΚ, οὕτως ἡ ΒΓ πρὸς ΓΚ, καὶ ὡς ἄρα ἡ ΜΑ πρὸς ΑΒ, οὕτως ἡ ΒΓ πρὸς ΓΚ. καὶ ἔστι τῆς μὲν ΑΒ ἡμίσεια ἡ ΑΔ, τῆς δὲ ΒΓ διπλῆ ἡ ΓΗ· ἔσται ἄρα καὶ ὡς ἡ ΜΑ πρὸς ΑΔ, οὕτως ἡ ΗΓ πρὸς ΚΓ. ἀλλ' ὡς ἡ ΗΓ πρὸς ΓΚ, οὕτως ἡ ΖΘ πρὸς ΘΚ διὰ τὰς παραλλήλους τὰς ΗΖ, ΓΘ· καὶ συνθέντι ἄρα ὡς ἡ ΜΔ πρὸς ΔΑ, ἡ ΖΚ πρὸς ΚΘ. ἴση δὲ ὑπόκειται καὶ ἡ ΑΔ τῇ ΘΚ, ἐπεὶ[1] καὶ τῇ ΓΖ ἴση ἐστὶν ἡ ΑΔ[1]· ἴση ἄρα καὶ ἡ ΜΔ τῇ ΖΚ· ἴσον ἄρα καὶ τὸ ἀπὸ ΜΔ τῷ ἀπὸ ΖΚ. καὶ ἔστι τῷ μὲν ἀπὸ ΜΔ ἴσον τὸ ὑπὸ ΒΜΑ μετὰ τοῦ ἀπὸ ΔΑ, τῷ δὲ

[1] ἐπεὶ . . . ΑΔ. Hultsch thinks these words are interpolated; they appear in both other versions.

Δ, E respectively, and let ΔΛ be joined and produced, and let it meet ΓB produced in H, and let EZ be drawn at right angles to BΓ in such a way that ΓZ is equal to AΔ, and let ZH be joined and parallel to it let ΓΘ be drawn, and, since the angle KΓΘ is given, from the given point Z let ZΘK be so drawn as to make ΘK equal to AΔ or to ΓZ (that this is possible is proved by the cochloidal line), and let KΛ be joined and produced, and let it meet AB produced in M; I say that $\Lambda\Gamma : \mathrm{K}\Gamma = \mathrm{K}\Gamma : \mathrm{MA} = \mathrm{MA} : \mathrm{A}\Lambda$.

Since BΓ is bisected at E and KΓ lies in BΓ produced, therefore

$$\mathrm{BK}\,.\,\mathrm{K}\Gamma + \Gamma\mathrm{E}^2 = \mathrm{EK}^2. \qquad \text{[Eucl. ii. 6}$$

Let EZ^2 be added to both sides.

Therefore $\mathrm{BK}\,.\,\mathrm{K}\Gamma + \Gamma\mathrm{E}^2 + \mathrm{EZ}^2 = \mathrm{EK}^2 + \mathrm{EZ}^2$,

that is $\mathrm{BK}\,.\,\mathrm{K}\Gamma + \Gamma\mathrm{Z}^2 = \mathrm{KZ}^2$. [Eucl. i. 47

And since $\mathrm{MA} : \mathrm{AB} = \mathrm{M}\Lambda : \Lambda\mathrm{K}$

and $\mathrm{M}\Lambda : \Lambda\mathrm{K} = \mathrm{B}\Gamma : \Gamma\mathrm{K}$,

therefore $\mathrm{MA} : \mathrm{AB} = \mathrm{B}\Gamma : \Gamma\mathrm{K}$.

And $\mathrm{A}\Delta = \frac{1}{2}\mathrm{AB}$, $\Gamma\mathrm{H} = 2\mathrm{B}\Gamma$.

Therefore $\mathrm{MA} : \mathrm{A}\Delta = \mathrm{H}\Gamma : \mathrm{K}\Gamma$.

But on account of HZ, ΓΘ being parallels,

$$\mathrm{H}\Gamma : \Gamma\mathrm{K} = \mathrm{Z}\Theta : \Theta\mathrm{K}.$$

Therefore, compounding,

$$\mathrm{M}\Delta : \Delta\mathrm{A} = \mathrm{ZK} : \mathrm{K}\Theta.$$

But by hypothesis $\mathrm{A}\Delta = \Theta\mathrm{K}$, since $\Gamma\mathrm{Z} = \mathrm{A}\Delta$;

therefore $\mathrm{M}\Delta = \mathrm{ZK}$;

therefore $\mathrm{M}\Delta^2 = \mathrm{ZK}^2$.

And $\mathrm{M}\Delta^2 = \mathrm{BM}\,.\,\mathrm{MA} + \Delta\mathrm{A}^2$ [Eucl. ii. 6

ἀπὸ ZK ἴσον ἐδείχθη τὸ ὑπὸ BKΓ μετὰ τοῦ ἀπὸ ZΓ, ὧν τὸ ἀπὸ ΑΔ ἴσον τῷ ἀπὸ ΓZ (ἴση γὰρ ὑπόκειται ἡ ΑΔ τῇ ΓZ)· ἴσον ἄρα καὶ τὸ ὑπὸ BMA τῷ ὑπὸ BKΓ· ὡς ἄρα ἡ MB πρὸς BK, ἡ ΓK πρὸς MA. ἀλλ' ὡς ἡ BM πρὸς BK, ἡ ΛΓ πρὸς ΓK· ὡς ἄρα ἡ ΛΓ πρὸς ΓK, ἡ ΓK πρὸς AM. ἔστι δὲ καὶ ὡς ἡ MB πρὸς BK, ἡ MA πρὸς ΑΛ· καὶ ὡς ἄρα ἡ ΛΓ πρὸς ΓK, ἡ ΓK πρὸς AM, καὶ ἡ AM πρὸς ΑΛ.

2. SQUARING OF THE CIRCLE

(a) General

Plut. *De Exil.* 17, 607E, F

'Ανθρώπου δ' οὐδεὶς ἀφαιρεῖται τόπος εὐδαιμονίαν, ὥσπερ οὐδ' ἀρετὴν οὐδὲ φρόνησιν. ἀλλ' 'Αναξαγόρας μὲν ἐν τῷ δεσμωτηρίῳ τὸν τοῦ κύκλου τετραγωνισμὸν ἔγραφε.

Aristoph. *Aves* 1001-1005

METΩN. Προσθεὶς οὖν ἐγὼ
τὸν κανόν' ἄνωθεν τουτονὶ τὸν καμπύλον,
ἐνθεὶς διαβήτην—μανθάνεις; ΠΕΙΣΘΕΤΑΙΡΟΣ.
οὐ μανθάνω.
METΩN. 'Ορθῷ μετρήσω κανόνι προστιθείς, ἵνα
ὁ κύκλος γένηταί σοι τετράγωνος.

[a] This reference shows the popularity of the problem of squaring the circle in 414 B.C., when the *Birds* was first produced. Meton, who is here burlesqued, is the great astronomer who about eighteen years earlier had found that after any period of 6940 days (a little over nineteen solar years) the sun and moon occupy the same relative positions as at the beginning, and had just built a water-clock worked by water from a neighbouring spring on the Colonus in the Athenian Agora. Actually, Meton made no contribution to squaring the circle; all he seems to be represented as doing is to divide the circle into four quadrants by two diameters at right angles.

and it was proved that

$$ZK^2 = BK \cdot K\Gamma + Z\Gamma^2,$$

and here $\Gamma Z^2 = A\Delta^2$ (for by hypothesis $A\Delta = \Gamma Z$);
therefore $BM \cdot MA = BK \cdot K\Gamma$;
therefore $MB : BK = \Gamma K : MA$. [Eucl. vi. 16
But $BM : BK = \Lambda\Gamma : \Gamma K$;
therefore $\Lambda\Gamma : \Gamma K = \Gamma K : AM$.
And $MB : BK = MA : A\Lambda$;
and therefore $\Lambda\Gamma : \Gamma K = \Gamma K : AM = AM : A\Lambda$.

2. SQUARING OF THE CIRCLE

(a) General

Plutarch, *On Exile* 17, 607E, F

There is no place that can take away the happiness of a man, nor yet his virtue or wisdom. Anaxagoras, indeed, wrote on the squaring of the circle while in the prison.

Aristophanes, *Birds* 1001-1005 [a]

Meton. So then applying here my flexible rod, and there my compass—you understand? Peisthetairos. I don't.

Meton. With the straight rod I measure so that the circle may become a square for you.