MOTION AND STASIS IN GEOMETRY

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1. *Background*

Mark Schiefsky and I taught History of Science 206r *Geometry and Mechanics* in the Spring semester 2012. The url for its web-page is:

http://isites.harvard.edu/icb/icb.do?keyword=k83402

The main theme of our course was to investigate the distinction (perhaps even the mild antagonism) between Geometry as done in the style of Euclid and the tradition of mechanical problems in ancient Greek mathematics[1]. Among the topics we studied were early efforts at squaring the circle, Hippocrates of Chios' quadrature of lunes, the literature related to doubling the cube: "instrumental" solutions and the quadratrix. We read Archimedes' Measurement of the Circle, and his mechanical method related to quadrature; and ancient and modern commentary (including Descartes, Mach) on these issues.

We paid special attention to the ancient commentaries regarding neusis, or mechanical methods (or mathematical arguments related to motion), some of the commentarists having the sentiment that these methods are—somehow—not the same species as more othodox geometric arguments. Some goals we set ourselves in our discussions were:

- To learn the corpus of ancient mathematics that might be labeled as "neusis" and to figure out exactly what was intended—in the writings of the ancient commentarists—by emphasizing the distinction between this type of mathematical reasoning and any other type.
- To try to understand the source of this distinction (Plato's *The Republic* Book 6–see below— being one natural guess).
- To read a bit of Euclid's *Elements* as a contrast in style.

The web-site with the student presentations gives a very good sense of what is covered. We also found that the web-site of Henry Mendell with its wealth of related material was invaluable.

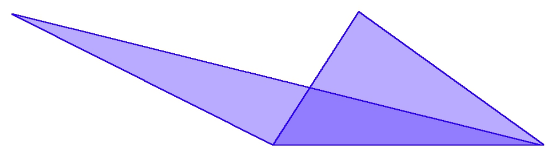
These rough notes are just small mathematical comments about a few specific issues that came up during the semester: Section 2 below gives a more detailed description of our starting question. Section 3 consists of a few comments about the hierarchy of algebraic

[1]It was a great privilege to learn from, and work with, Mark; and to have had the students we had, who gave wonderful presentations on the material.

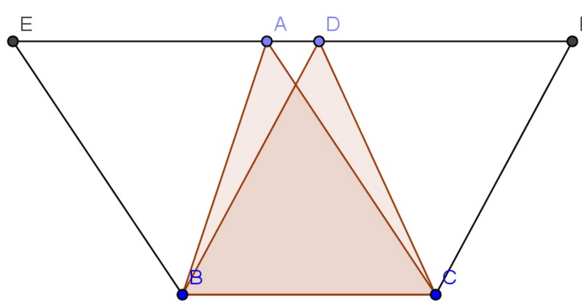
numbers and it relation to some of the classical ancient problems. Section 4 discusses some mathematics—not all—-behind Hippocrates's lunes. Section 5 is a reminder that our work suggests that the geometrization of motion is an interesting question to pursue. The final section discusses the question of "deriving" Archimedes' law of the lever, a question sparked by Ernst Mach's discussion of it in his *Science of Mechanics*.

2. The Motivating Question

The two shaded triangles in the figure below



have equal area. The ancient source that establishes this is Euclid's Proposition 37 of Book I of the *Elements* which says that triangles with the same base and height have the same area.



Proof: "Let ABC and DBC be triangles on the same base BC and in the same parallels AD and BC. I say that the triangle ABC equals the triangle DBC. Produce AD in both directions to E and F. Draw BE through B parallel to CA, and draw CF through C parallel to BD. (I.Post.2, I.31). Then each of the figures EBCA and

DBCF is a parallelogram, and they are equal, for they are on the same base BC and in the same parallels BC and EF. (I.35) Moreover the triangle ABC is half of the parallelogram EBCA, for the diameter AB bisects it. And the triangle DBC is half of the parallelogram DBCF, for the diameter DC bisects it. (I.34) Therefore the triangle ABC equals the triangle DBC. (*Common notions*)

Therefore *triangles which are on the same base and in the same parallels equal one another.* (QED)"

A more modern take on essentially the same mathematical issue is the proposition saying that *area* is invariant under certain types of transformations; specifically: *shear transformations* of the Euclidean plane[2].

The three-dimensional (modern) analogue of this same result is captured by the remark that no matter how you nudge a stack of coins, the volume of the stack is unchanged.

Here we have an example of vast difference between the ancient (Euclidean) and the modern approaches to very related mathematical truths.

In Euclid's *Elements* there is no palpable appearance of the concepts *transformation*, or *motion*; and even more remote would be the notions of *continuity*, or *perturbation*.

In contrast, much of modern geometry is expressed in the vocabulary of transformations, of symmetries, and often of *dynamical systems* and structures submitted to continuous change through time. With such concepts as backdrop, issues like stability (i.e., qualitative structures unperturbed by small changes) and instability emerge as well. *Variation* is often key; and even more: some mathematicians feel that for certain specific concepts, you don't

[2]a **shear transformation** is one that in Cartesian coordinates can be given by the formula $(x, y) \mapsto (x + f(y), y)$ where $f(y)$ is any reasonable function.

really understand them until you have understood the entire family of all possible variations of that concept.

Now, even if there is none of this vocabulary in Euclid, we can find quite a bit of continuous movement as an important theme in *other* ancient geometric texts, the "keyword" that gives the hint that movement plays a role is *mechanical*. Here, for example, is the opening of the pseudo-Aristotelian text *Mechanical problems*:

> Among things that occur according to nature (*kata phusin*), we wonder at those whose cause is unknown; among things that occur *para phusin,* we wonder at those that come about by means of art (*technê*) for the benefit of mankind. For in many cases nature acts in a way opposed to what is useful for us. For nature always acts in the same way and simply, while what is useful changes in many ways. Whenever, then, it is necessary to do something *para phusin*, because of the difficulty we are at a loss (*aporia*) and have need of art (*technê*). For this reason, we also call that part of art that assists in such situations (*aporiai*) a device (*mêchanê*). For as the poet Antiphon said, so it is:
>
> "By means of art we gain mastery (*kratoumen*) over things in which we are conquered by nature.
>
> Instances of this are those cases in which the lesser master (*kratein*) the greater, and things possessing a small inclination move great weights, and practically all those problems that we call mechanical. These are not entirely identical with physical problems, nor are they entirely separate from them, but they have a share in both mathematical and physical speculations: for the 'how' in them is made clear through mathematics, while the 'about what' is made clear through physics.

The *mechanical method* of Archimedes, as well, has quite different flavor from that of Euclid. Despite the fact that its connection with Physics might be more in the direction of *statics*—e.g., with *center of gravity* an Archimedean concept—it has a dynamic feel.

Could it be that Euclid's concentration on 'stasis' rather than movement is guided by the following sentiment, that one finds in Plato's *Republic* (527a,b)?

> "This at least, said I, "will not be disputed by those who have even a slight acquaintance with geometry, that this science is in direct contradiction with the language employed in it by its adepts." "How so?" he said. "Their language is most ludicrous, though they cannot help it, for they speak as if they were doing something and as if all their words were directed towards action. For all their talk

> is of squaring and applying and adding and the like, whereas in fact the real object of the entire study is pure knowledge." "That is absolutely true," he said. "And must we not agree on a further point?" "What?" "That it is the knowledge of that which always is, and not of a something which at some time comes into being and passes away." "That is readily admitted," he said, "for geometry is the knowledge of the eternally existent." "Then, my good friend, it would tend to draw the soul to truth, and would be productive of a philosophic attitude of mind, directing upward the faculties that now wrongly are turned earthward."

Whatever it is, the somewhat static nature of Euclid's *Elements* colors much of what happens in that text and has an especially strong effect on those aspects that are, of necessity, in some temporal sequence, or in motion; namely: *constructions* (*kataskeue*). Euclid constructs things all the time, extending lines, bisecting line segments, drawing lines through a point parallel to another line, drawing circles, etc. But the tools of construction are exquisitely limited[3]. Moreover, since all the so-called *classical problems* are problems of construction, the rules prescribed for their construction are similarly austere. With all this, however, you find that outside the Euclidean corpus, the rules for construction allow for somewhat greater flexibility. Indeed, it is only with modern mathematics that one comes to see that certain austere constructions are impossible, and are achievable only with the aid of certain interesting non-Euclidean tools.

What we have just described is the main theme of our course, and gives hints of the texts we will be reading. One focus of our course is precisely the contrast between the *Euclidean* and the *mechanical* or *dynamic* elements in the corpus of ancient mathematics. We will also distinguish concepts related to 'motion'—such as continuity, divisibility, and passage from the *greater* to the *lesser*[4]—and how these issue are handled in ancient mathematics.

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[3]In an article *Mathematics and Narrative: An Aristotelian Perspective* by G.E.R. Lloyd that will soon be published, Lloyd comments as follows on the quotation from Plato's *Republic* we have given above:

> So it is ridiculous to talk of squaring a circle, for instance, when nothing is done to the circle. Plato wishes to recommend mathematics for the education of those who are to rule in his ideal city, and that is because it provides training in abstract thought... Later Platonists, convinced, no doubt, that they were being loyal to the masters own thought on the subject, developed the criticism further, suggesting that any talk of moving mathematical figures, or otherwise applying physical or mechanical concepts to them, should be banned.

[4]fairly explicit versions of an 'intermediate value theorem,' such as in Plato's *Parmenides*:

> "But greatness and smallness are constituents of inequality." "Yes." "Then the one, such as we are discussing, possesses greatness and smallness?" "So it appears." "Now surely greatness and smallness always keep apart from one another." "Certainly." "Then there is always something between them." "There is." "Can you think of anything between them except equality?" "No, only equality." "Then anything which has greatness and smallness has also equality, which is between the two." "That is clear."

3. Constructible numbers, solvable numbers, other numbers

One theme of our course had to do with the question of whether or not it is *natural* to restrict constructions in plane geometry to be—as in Euclid—via ruler and compass alone. (My vote is that it is not terribly natural.) The effect of such ruler-and-compass constructions (iterated), when starting with a given unit (which we can identify with the number 1) is to provide lengths whose ratios to the unit are square roots of rational expressions of numbers that are square roots of rational expressions of numbers that are square roots of ... and so on ... ending with rational numbers. It is traditional to call such proportions *constructible.* Modern algebra often isolates a much larger category of numbers it considered to be in some slightly more liberal sense primitively constructible; these could be called *solvable numbers* and can be described very much like the class of constructible numbers but as basic operation one allows oneself to take n-th roots (for arbitrary choices of n rather than only square roots) for each iteration.

In contrast to the construction of these so-called constructible numbers which rely on the instruments of ruler and compass alone, to achieve the construction of the larger class of solvable numbers requires some device that can actually extract n-th roots; or equivalently, can construct two or more *mean proportionals* to any given pair of lengths, as in Alma Steingart's class presentation of the constructions described by Eutocius. We have the hierarchy:

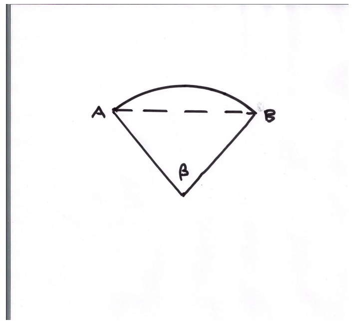
$$\textit{Rational} \subset \textit{Constructible} \subset \textit{Solvable} \subset \textit{Algebraic} \subset \textit{All}$$

That there are numbers that are roots of fifth degree polynomial equations with whole number coefficients (and hence algebraic numbers) that are *not* solvable in the sense just described is one of the great discoveries of 19th century algebra.

Many of the classical Greek problems relate to this hierarchy of constructibility. For example, the question of *duplication of the cube* depends directly on the question of whether the cube root of 2 (which is evidently 'solvable') is constructible (it is not). Similarly, the question of quadrature of the circle relates to the question of whether π is constructible It is not; in fact, it isn't even algebraic. (I don't know of any proof of the non-constructibility of π that doesn't, as well, show that it isn't algebraic.)

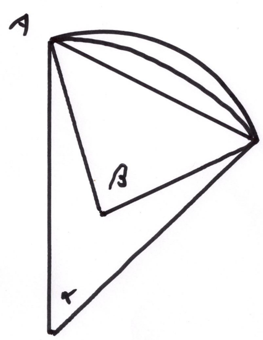
4. What is a lune?

First, suppose that we have an arc of a circle of radius s, cutting out an angle $0 < \beta < \pi$ from its center, and subsuming a chord AB as in this figure.



Then we have some equations:

(1) **Length:** The length of the chord AB is $2s\sin(\beta/2)$.
Note that—up to scale—a **lune** is given by two angles $0 < \alpha \leq \beta < \pi$, in circles of radius r and s respectively, cutting out the same chord AB.



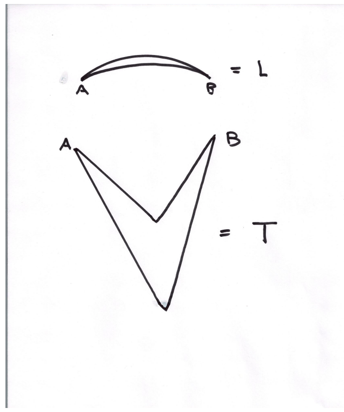
This latter requirement gives us (and in fact, is equivalent to) the equation

$$r\sin(\alpha/2) = s\sin(\beta/2). \quad (\mathbf{1})$$

(2) **Area:** The area of the entire pie-shaped wedge of the in the first figure is equal to βs^2, and the area of the second pie-shaped wedge subsuming angle α is equal to αr^2. The difference between these two areas, i.e.,

$$\beta s^2 - \alpha r^2$$

is simply $L - T$.



SO, if (... if ...) you want to perform the most evident quadrature (by arranging L to be equal to T) you would look for pairs α, β such that

$$(\mathbf{2}) \qquad \beta s^2 = \alpha r^2.$$

Putting (**1**) and (**2**) together you see that you are looking for pairs $\alpha = 2x, \beta = 2y$ that satisfy the single equation:

$$(\mathbf{3}) \qquad sin^2(x)/x = sin^2(y)/y.$$

In the range we are working, the function $\sin^2(x)/x$ has a single maximum (at the value $x = \gamma$ where $\gamma \tan(\gamma) = 2$. This γ is near $\pi/2$. Also, $\sin^2(x)/x$ vanishes at 0 and π. So for *every* α, there is a unique β in the range such that $(x, y) = (\alpha, \beta)$ is a solution of equation **(3)**. Moreover, for *every positive real number* n there is a *unique* pair $0 < \alpha \leq \beta < \pi$ such that $n := \beta/\alpha$ and the above equation holds. We get a curious function:

$$N(\alpha) := \beta/\alpha$$

where the "β" is the unique "β" such that $(x, y) = (\alpha, \beta)$ solves our equation **(3)**. The interesting values, though, seem to correspond to cases when $N(\alpha) = n$ is an integer or a near integer.

(1) It may be surprising that this *most evident quadrature* meshes with the somewhat *different* quadrature (closer to the texts!) related to polygons of Hippocrates, when x and y are solutions that have whole number (or near-whole-number) ratios. In the special cases where, in fact, $N(\alpha) = n \geq 2$ *is* an integer, we have the diagram generalizing some of the actual constructions of Hippocrates:

Put figure here

In question, again, are (the areas of) two figures:

(a) The polygon of n sides that has two properties:
   (i) it is inscribed in a circle,
   (ii) all but one side (which I will call the "long side") has the same length.

For continuity (i.e., "mechanical") reasons for each integer $n \geq 2$, there exists a continuum of such polygons (up to similarity equivalence). Let's denote these by P. One mode of quadrature is to put a segment on the long side that is *similar to* the segment cut out by the circumscribed circle and any of the n short sides. This produces the lune, call it $\mathcal{L}$. Call the angle subsumed by any of the short sides α; put $\beta := n\alpha$. The condition that $\mathcal{L} = P$ is given by exactly the same formula **(3)** above:

$$\sin^2(\beta)/\beta = \sin^2(\alpha)/\alpha.$$

The questions:

(i) Whenever we have an α with $N(\alpha)$ and integer, we get both a T and a P as above with the same area. Is there a neat geometric construction that negotiates from one of them to the other?

(ii) what are the constructible quadraturable lunes? I.e., which are the α's such that all the diagrams discussed above can be constructed by Euclidean means?

5. Galois-theoretic questions raised by Hippocrates' lunes

Following up the properties of $\frac{\sin^2(x)}{x}$, let's denote by (p, q) the lune built out of p arcs of one of the circles against q arcs of the other. This leads to the rational function:

$$\textbf{(1)} \quad R_{p,q}(z) := \quad q(z^p + z^{-p}) + 2(p-q) - p(z^q + z^{-q})$$

invariant under the transformation $z \mapsto 1/z$, and such that

$$R_{p,q}(z) \quad = \quad -R_{q,p}(z),$$

or—assuming that $p \geq q$— we get the polynomial:

$$\textbf{(2)} \quad F_{p,q}(z) \;=\; z^p R_{p,q}(z) = \quad qz^{2p} - pz^{p+q} + 2(p-q)z^p - pz^{p-q} + q.$$

The issue is whether or not (all) the roots of the polynomial **(2)** are "constructible" via the standard algebraic operations plus square roots, starting from the rational numbers. If so, say that (p, q) *is constructible*, although to really know what this concept of constructibility entails requires discussion. One might wonder, while one is at it, whether these roots are all solvable, or not.

To investigate this in a more detailed manner, it is natural ask for the Galois group of the polynomial **(2)** (more specifically: the Galois group of the Galois closure of the splitting field of this polynomial over the rational field). I haven't really answered this question this in full generality.

But in anticipation of an eventual full account, it might be good to record some elementary observations:

(1) We have that

$$z^{2p} \cdot F_{p,q}(1/z) \;=\; F_{p,q}(z) \;=\; -z^{p-q}F_{q,p}(z)$$

so if $z = \theta$ is a root then so is $z = \theta^{-1}$

(2) $z = 1$ is a root of **(1)**; in fact, it is a double root.

(3) Note also that the coefficients of the polynomial **(1)** are all linear in p and q, so if p and q are not relatively prime, i.e., $p = dp_o$, and $q = dq_o$ with d the greatest common divisor of p and q, then dividing **(1)** by d we get the polynomial relation:

$$(\mathbf{3}) \quad F_{p,q}(z) \;=\; d \cdot F_{p0,q_o}(z^d)$$

so to determine the Galois structure of the polynomial **(2)** we may concentrate on pairs p, q that are relatively prime.

Some of the modern commentaries (Steele's, among others) mention that if p is prime and q is not divisible by p, then (p, q) is not constructible unless p is of the form $2^k + 1$. By (1) above, the same is true if q is a prime not of the form $2^k + 1$ and p is not divisible by p. Here is a proof of this using some algebra (the "Eisenstein criterion"). Make the substitution $z = y - 1$ and rewrite the above polynomial in terms of y:

$$F(y) := q(y+1)^{2p} - p(y+1)^{p+q} + 2(p-q)(y+1)^p - p(y+1)^{p-q} + q,$$

so that $y = 0$ is a root of this equation. Let's compute the coefficient of the linear term:

$$q \cdot 2p - p(p+q) + 2(p-q)p - p(p-q) = 0,$$

so $F(y)$ is divisible by y^2; i.e.

$$F(y) = y^2 \cdot G(y)$$

where $G(y)$ is a polynomial of degree $2p - 2$.

and the degree two term of $F(y)$ is:

$$\frac{1}{2}\{q \cdot 2p(2p-1) - p(p+q)(p+q-1) + 2(p-q)p(p-1) - p(p-q)(p-q-1)\} = -2pq^2 + 2p^2q.$$

Now let us assume that p is a prime, noting that (since q and p are relatively prime) the constant term of $G(y)$ is divisible by p but not by p^2.

Putting at the end of the equation all the terms divisible by p (I do this for no good reason, but it makes a certain claim I'll make a bit more visible): we write:

$$F(y) = q \cdot \{(y^2 + 2y + 1)^p - 2(y+1)^p + 1\} - p \cdot \{(y+1)^{p+q} - 2(y+1)^p + (y+1)^{p-q}\},$$

and we'll deal with each bracketed piece.

Then all the coefficients of the polynomial $F(y)$ except for that of the leading term (i.e., qy^{2p}) are divisible by p. Hence this is also true for $G(y)$ and the coefficient of the constant term of $G(y)$ is divisible by p but *not* by p^2. These are the requirements of the *Eisenstein criterion* which then guarantees that this polynomial $G(y)$ (and therefore also **(1)** when deprived of its double root $z = 1$) is irreducible.

For reasons that we can discuss, this means that the roots of **(1)** are *not constructible* if $2p - 2$ is not a power of two; equivalently, if p is not of the form $2^k + 1$. I'm not clear, yet, about what this really implies if p is of the form $2^k + 1$.

6. Ammonius and the Long Line

Bryson, by report, has one of the earlier (and *crude*) attempts at squaring the circle. Amy Koenig gave a very illuminating presentation of this material; see her notes on the web-page of our course.

We have various descriptions (due to Themistius, Alexander of Aphrodisias, and John Philoponus commenting on Aristotle's *Posterior Analytics*) of Bryson's putative solution of the *squaring of the circle*. Here it is taken from John Philoponus (transl. Mendell, Schiefsky)

> The circle (Bryson says) is larger than any inscribed rectilinear figure but is less than every circumscribed one... but also the rectilinear figure drawn between the circumscribed and inscribed figure is smaller than the circumscribed and larger than the inscribed figure. **Things larger and smaller than the same are equal to one another**. Therefore the circle is equal to the rectilinear figure drawn between the inscribed figure and the circumscribed figure. But we can construct a square equal to any rectilinear figure. Therefore it is possible to produce a square equal to a circle.

One way of reconstruing this crude formulation is to imagine each of the inscribed and circumscribed rectilinear figures discussed as turned into a square of equal size. Then one is just dealing with an increasing zoom-family of squares. A basic geometric version of what a modern might call the *intermediate value theorem* would be the statement that if you start with a square that is smaller than the circle and expand continuously to achieve a square larger, there must have been an intermediate square of exactly the same size as the circle, so QED; rather the way a stopped clock is correct twice a day.

Ammonius (a 5th century neoplatonist) raised an interesting subtle objection to this (as reported by John Philoponus). Namely Ammonius was bothered by the fact that Bryson is comparing and lumping together geometric figures of different types, sorted by size. He offered the ingenious example of *horn-angles* as a cautionary tale warning against too blithe a claim about equality (using what a modern might label an *intermediate value theorem* argument) when you are talking about geometric figures of various types. Namely, Ammonius ranges together straight angles and angles made at the intersection of a straight line and a circle and orders them in terms of “increasing size” (in a way that is very natural).

What struck me as interesting is that Ammonius seems to have predated by a millenium and a half, a very modern construction that has served as one of the neat counter-examples to many many things in elementary topology that one might otherwise imagine were true. The counter-example is usually referred to as **the long line** and it stands in the subject as a monstrosity, but a very useful one enamored by the early twentieth century school of Polish topologists.

Here is how this 'long line' is put together: to every real number x, you associate an infinite closed half line $L_x := [0, +\infty) = \{r \geq 0\}$ and let $\mathcal{L}$ (our *long line*) be the union of all these L_x's. A point in $\mathcal{L}$ has two “coordinates” (x, r) where x is any real number and r is a non-negative real number. You can define an *order relation* on this long line by stipulating that $(x', r') > (x, r)$ if either $x' > x$; or $x' = x$ and $r' > r$. This gives a total ordering on the line and it isn't difficult to describe the natural topology that goes along with it. Now any point (x, r) of $\mathcal{L}$ can be associated to one of Ammonius's horn-angles by the following rule: if $r = 0$ you are dealing with a straight-line-angle with vertex at the intersection of the horizontal line in the plane with a straight line of slope x; if $r > 0$ you are dealing with a horn-angle with vertex at the intersection of the horizontal line in the plane with a circle of radius $1/r$ that has slope x at the vertex.

To sum up, the modern point set topologist's *long line* is in a very straightforward way pre-envisioned, and beautifully geometrized, by Ammonius with his array of horn and straight angles. Both in its ancient formulation and its modern, it was constructed to provide counter-examples.

7. Time geometrized

In some of the course presentations (including those of Connemara Doran, and Juan Marin) various 'moving machines' situated in the plane were described. In the presentation of Stephanie Dick, the predominant feature was three-dimensional geometry, rather than planar geometry and there was less dependence on motion. Of course, 'motion' in planar geometry can be represented statically as 'a configuration' in solid geometry. Nowadays, we are very happy with motion as being described statically, and space-time as a single thing. Exactly when and how did *geometrization of time* become natural? Certainly by the time of Oresme. As we were working through this material I felt that this *geometrization of time* is a theme I want to know more about.

8. Archimedes: Equilibrium of the plane, Book I

Archimedes' text is poised between physics and mathematics and it is difficult to categorize it.

(1) Is it *pure mathematics*? (where, for example, the concepts baros, equilibrium, etc. are being given their "starter axioms") (2) Is it *physics*? (where the mathematics merely has a descriptive, rather than ontological, purpose) (3) Is it some curious combination of mathematics and physics? (where the concepts are on the road to being imported into math, starting their lives in physics, the way—say—Maxwell's equations move into Riemannian geometry)

If (1), then the issue of circularity is gone: all of math is circular and that's a good thing. If (2), then—taking Mach's viewpoint of physical law as as being (nothing more than) the most economical way of expressing the sum total of our observational experience—hey, if your experiments were specifically designed to verify Archimedes' postulates, and you want to know whether they are also evidence for his law of the lever, well, you have to use his derivation of the law of the lever from the postulates to see this; if, however, you experiments were designed to check the law of the lever, then running his derivation backwards gives you evidence for his postulates. What's the complaint? (Answer: some inner intent.) If (3) then one is in that middle-zone, and the derivations are even more useful to convince you of some inner coherence of the set-up.

Here is a proof—perhaps *the* only proof[5] of Archimedes' *Law-of-the-lever* based on the postulates of Archimedes that Archimedes' law-of-the-lever holds.

Imagine a homogeneous bar of length $2m + 2n$, placed symmetrically on a balance line. So a "weight" $m + n$ is distributed to the left of the fulcrum, and an equal weight is distributed to the right. Now make a laser-like slicing of the bar, splitting it into two bars of weights $2m$ (say, on the left) and $2n$ (on the right). Replace each of those bars by single weights $2m$ and $2n$ concentrated at their respective centers of gravity. Then the concentrated weight weighing $w := 2m$ is of distance $d := n$ to the left of the fulcrum while the concentrated weight weighing $w' := 2n$ is of distance $d' := m$ to the left of the fulcrum. By the postulates of Archimedes, we get that this is in equilibrium.

Note that:

$$(*) \quad w \cdot d \;=\; w' \cdot d'.$$

Using further postulates of Archimedes, the two weights w, w' on either side of a fulcrum and of distances d, d' (respectively) to the fulcrum are in equilibrium if and only if (*) holds. This is his Law-of-the-lever.

Now this is 'proved' by Archimedes, subject to his postulates. Mach has interesting comments about this proof, one of which I will paraphrase, putting it in slightly more mathematical terms than he does, but without entirely violating his intent, I think.

(1) Some *unstated* pre-postulates are required to hold, before one can even begin to make the above kind of argument. For example,

[5] given with minor modifications in many of our readings

- There *is* a property that determines equilibrium: for example, there a single function

$$F(\ldots \text{relevant variables} \ldots)$$

such that if the function F evaluated on the state of the left-of-the-fulcrum part of the balance bar is equal to the function F evaluated on the state of the right-of-the-fulcrum part of the balance bar, then (*and only then*) is equilibrium achieved.
- The "*relevant variables*" referred to above are just weight w and distance-to-fulcrum d (and not, say, dependence on the position of the observer, or color of the balance bar). So our function F may be thought of as a two-variable function $F(w, d)$.

(2) The stated postulates are the ones at the beginning of Archimedes' treatise.

Now Mach said something in the initial edition of *Science of Mechanics* that stirred up quite a debate, and deserves some discussion. The issue has to do with the fact that

- the statement that the function $F(w, d)$ is linear in both variables, and therefore—up to scaling— $F(w, d) = w \cdot d$ is equivalent to Archimedes' law of the lever, and
- this linearity follows from the axioms, and
- yet Mach worries that people think they are "making up properties of nature with the help of self-evident suppositions." (See page 27; also Page 19 where Mach curiously uses the word "covert").

This has inspired much argument; recently Palmieri. My feeling about Palmieri is that he is very heavy-handed in his derivations, so here is what is a more direct derivation of bilinearity of $F(w, d)$.

Given Archimedes' axioms we get that

$$(\mathbf{2}) \quad F(w_1 + w_2, d) \ = \ F(w_1, d) \ + \ F(w_2, d),$$

(so it is linear in the first variable) and:

$$(\mathbf{3}) \quad F(2w, d) \ = \ F(w, d - \Delta) \ + \ F(w, d + \Delta).$$

Putting **(2)** and **(3)** together we can write

$$(\mathbf{4}) \quad F(2w, d) \ = \ F(w, d) + F(w, d) \ = \ F(w, d - \Delta) \ + \ F(w, d + \Delta).$$

So

$$(\mathbf{5}) \quad F(w, d + \Delta) - F(w, d) \ = \ F(w, d) - F(w, d - \Delta),$$

or changing notation to make it easier to look at: for any fixed w put $f(x) = F(w, x)$ and set $a := d - \Delta$ and $b := \Delta$, so we have for *any* $a, b \geq 0$:

$$(\mathbf{6}) \quad f(a + b) - f(a) = f(a + 2b) - f(a + b)$$

and iterating this as many times as we want, we get:

$$(\mathbf{7}) \quad f(a+b) - f(a) = f(a+2b) - f(a+b) = f(a+3b) - f(a+2b) = \ldots$$

or, since our f vanishes at 0, we get (taking $a = 0$) that

$$(*) \quad f(nb) = nf(b)$$

for any b and positive integer n. Or, putting $b = c/m$ we can also write this as $f(c/m) = 1/mf(c)$ for any c and positive whole number m, and putting this together by taking c to be nx for any $x > 0$ and any positive whole number n we get $f(nx/m) = nf(x/m) = n/m \cdot f(x)$; i.e.,

$$(\mathbf{8}) \quad f(rx) = rf(x)$$

for any x and positive rational number r. And so (at least under the assumption that f is continuous[6]) f is linear.

Why does Mach think that there is something circular in this? We had the pleasure of listening to Gerald Holton, who offered some illuminating background for this issue.

[6]This is a it necessary assumption; it is false without it.