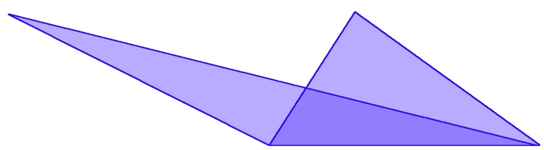
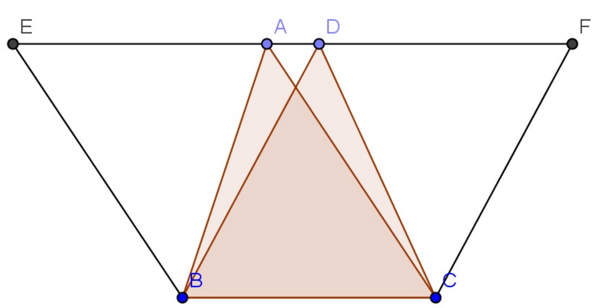
MOTION AND STASIS IN GEOMETRY

Some comments on the background for our course

The two shaded triangles in the figure below



have equal area. The ancient source that establishes this is Euclid's Proposition 37 of Book I of the *Elements* which says that triangles with the same base and height have the same area.



Proof: "Let ABC and DBC be triangles on the same base BC and in the same parallels AD and BC. I say that the triangle ABC equals the triangle DBC. Produce AD in both directions to E and F. Draw BE through B parallel to CA, and draw CF through C parallel to BD. (I.Post.2, I.31). Then each of the figures EBCA and

> DBCF is a parallelogram, and they are equal, for they are on the same base BC and in the same parallels BC and EF. (I.35) Moreover the triangle ABC is half of the parallelogram EBCA, for the diameter AB bisects it. And the triangle DBC is half of the parallelogram DBCF, for the diameter DC bisects it. (I.34) Therefore the triangle ABC equals the triangle DBC. (*Common notions*)
>
> Therefore *triangles which are on the same base and in the same parallels equal one another.* (QED)"

A more modern take on essentially the same mathematical issue is the proposition saying that *area* is invariant under certain types of transformations; specifically: *shear transformations* of the Euclidean plane[1].

The three-dimensional (modern) analogue of this same result is captured by the remark that no matter how you nudge a stack of coins, the volume of the stack is unchanged.

Here we have an example of vast difference between the ancient (Euclidean) and the modern approaches to very related mathematical truths.

In Euclid's *Elements* there is no palpable appearance of the concepts *transformation*, or *motion*; and even more remote would be the notions of *continuity*, or *perturbation*.

In contrast, much of modern geometry is expressed in the vocabulary of transformations, of symmetries, and often of *dynamical systems* and structures submitted to continuous change through time. With such concepts as backdrop, issues like stability (i.e., qualitative structures unperturbed by small changes) and instability emerge as well. *Variation* is often key; and even more: some mathematicians feel that for certain specific concepts, you don't

[1]a **shear transformation** is one that in Cartesian coordinates can be given by the formula $(x, y) \mapsto (x + f(y), y)$ where $f(y)$ is any reasonable function.

really understand them until you have understood the entire family of all possible variations of that concept.

Now, even if there is none of this vocabulary in Euclid, we can find quite a bit of continuous movement as an important theme in *other* ancient geometric texts, the “keyword” that gives the hint that movement plays a role is *mechanical*. Here, for example, is the opening of the pseudo-Aristotelian text *Mechanical problems*:

> *One marvels at things that happen according to nature, to the extent the cause is unknown, and at things happening contrary to nature, done through art for the advantage of humanity. Nature, so far as our benefit is concerned, often works just the opposite to it. For nature always has the same bent, simple, while use gets complex. So whenever it is necessary to do something counter to nature, it presents perplexity on account of the difficulty, and art [techne] is required. We call that part of art solving such perplexity a mechane. As the poet Antiphon puts it:*
>
> > *We win through art where we are beaten through nature.*
>
> *Such it is where the lesser overcomes the greater, and when things having little impetus move great weights. And we term this entire class of problems mechanics. Mechanics isnt just restricted to physical problems, but is common alike to the theorems of mathematics as well as physics: the how is clear through mathematics, the what is clear through physics.*

The *mechanical method* of Archimedes, as well, has quite different flavor from that of Euclid. Despite the fact that its connection with Physics might be more in the direction of *statics*—e.g., with *center of gravity* an Archimedean concept—it has a dynamic feel.

Could it be that Euclid’s concentration on ’stasis’ rather than movement is guided by the following sentiment, that one finds in Plato’s *Republic* (527a,b)?

> *“This at least, said I, “will not be disputed by those who have even a slight acquaintance with geometry, that this science is in direct contradiction with the language employed in it by its adepts.” “How so? he said. “Their language is most ludicrous, though they cannot help it, for they speak as if they were doing something and as if all their words were directed towards action. For all their talk is of squaring and applying and adding and the like, whereas in fact the real object of the entire study is pure knowledge.” “That is absolutely true, he said. “And must we not agree on a further point? “What? “That it is the knowledge of that which always is, and not of a something which at some time comes into being and passes away.*

> *"That is readily admitted, he said, "for geometry is the knowledge of the eternally existent. "Then, my good friend, it would tend to draw the soul to truth, and would be productive of a philosophic attitude of mind, directing upward the faculties that now wrongly are turned earthward.*

Whatever it is, the somewhat static nature of Euclid's *Elements* colors much of what happens in that text and has an especially strong effect on those aspects that are, of necessity, in some temporal sequence, or in motion; namely: *constructions.* Euclid constructs things all the time, extending lines, bisecting line segments, drawing lines through a point parallel to another line, drawing circles, etc. But the tools of construction are exquisitely limited. Moreover, since all the so-called *classical problems* are problems of construction, the rules prescribed for their construction are similarly austere. With all this, however, you find that outside the Euclidean corpus, the rules for construction allow for somewhat greater flexibility. Indeed, it is only with modern mathematics that one comes to see that certain austere constructions are impossible, and are achievable only with the aid of certain interesting non-Euclidean tools.

What we have just described is the theme of our course, and gives hints of the texts we will be reading. One focus of our course is precisely the contrast between the *Euclidean* and the *mechanical* or *dynamic* elements in the corpus of ancient mathematics.