Sources and Studies
in the History of Mathematics and
Physical Sciences

Pappus of Alexandria:
Book 4 of the *Collection*

Edited With Translation and Commentary by

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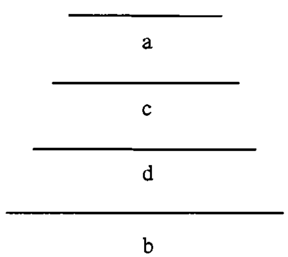
so <is> ZT to TK, on account of the parallels HZ and CT.[1] And therefore, componendo: as MD <is> to DA, <so is> ZK to KT.[2] However, AD has also been posited as equal to TK.[3] Therefore, MD is equal to ZK[4] as well. Therefore, the square over MD is equal to the square over ZK, also. And the rectangle BMA, taken together with the square over DA, is equal to the square over MD,[5] whereas the rectangle BKC taken together with the square over ZC has been shown to be equal to the square over ZK. Of these, the square over AD is equal to the square over CZ (for AD has been posited as equal to CZ). Therefore, the rectangle BMA is equal to the rectangle BKC, also. Therefore, as MB <is> to BK, <so is> CK to MA.[6] But as BM <is> to BK, <so is> LC to CK.[7] Therefore, as LC <is> to CK, <so is> CK to AM. However, MA is to AL as MB <is> to BK, also.[8] And therefore, as LC <is> to CK, <so is> CK to AM, and <so is> AM to AL.

Prop. 25: Cube Duplication, Cube Construction in Given Ratio

#29 After this has been shown, it is very clear how one must, when a cube is *given*,[9] find another cube in a *given* ratio.



For:

Assume that the *given* ratio is that of the straight line a to the <straight line> b, and take c and d as two means in continuous proportion for a and b. Then the cube over a will be to the cube over c as a is to b. For this is clear from the *Elements*.[10]

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[1] VI, 2 ($\Delta HZK \sim \Delta CTK$ on parallel lines).

[2] V, 18.

[3] By construction (*neusis*).

[4] V, 9.

[5] II, 6.

[6] VI, 16.

[7] VI, 4 ($\Delta MBK \sim \Delta LCK$).

[8] VI, 4 ($\Delta MAL \sim \Delta MBK$).

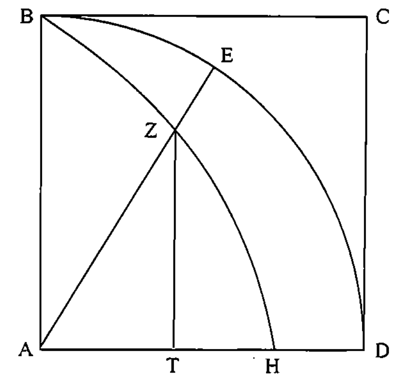
[9] Once again, note the occurrence of derivatives of the technical term δοθέν (250, 26, 27, and 28 Hu).

[10] a and c stand in the triple ratio of a:b (V, def. 11); cube numbers have two mean proportionals, and cube:cube = $(\text{side:side})^3$ (VIII, 12); the cubes with sides a and c stand in that same ratio (XI, 33).

Props. 26–29: Quadratrix[1]

Genesis and Symptoma of the Quadratrix

#30 For the squaring of the circle a certain line has been taken up by Dinostratus and Nicomedes[2] and some other more recent (mathematicians). It takes its name from the *symptoma* concerning it. For it is called "quadratrix" by them, and it has a *genesis* of the following sort.



Set out a square ABCD and describe the arc BED of a circle with center A, and assume that AB moves in such a way that while the point A remains in place, <the point> B travels along the arc BED, whereas BC follows along with the traveling point B[3] down the <straight line> BA, remaining parallel to AD throughout, and that in the same time both AB, moving uniformly, completes the angle BAD, i.e.: the point B <completes> the arc BED, and BC passes through the straight line BA, i.e.: the point B travels down BA.[4] Clearly it will come to pass that both AB and

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[1] The Latin word "quadratrix" (i.e., squaring line) translates the Greek name (τετραγωνίζουσα) for the transcendent curve that will be the subject of Props. 26–29. The Latin version is commonly used as the standard name for this particular curve, though the term can have other meanings, too.

[2] The common author Nicomedes connects the passages on the conchoid and quadratrix curves. Dinostratus was a late fourth century BC mathematician, the brother of Menaechmus, who invented the conics as locus curves. On the authorship concerning the curve quadratrix and its *symptoma*-mathematics see the commentary.

[3] συνακολουθείτω; the basic verb is, once again "ἀκολουθέω" = follow along in order. As in the other instances in *Coll.* IV, it does not have the connotation of strict logical derivation – on the contrary (see below). On the use of "ἀκολουθεῖν" compare the remarks on analysis-synthesis in the introduction to Props. 4–12.

[4] This generation via synchronized motions is reminiscent of the *genesis* of the spiral in Prop. 19; the connection between these two curves has been emphasized by Knorr (e.g., Knorr 1978a, 1986).

BC reach the straight line AD at the same time. Now, while a motion of this kind is taking place, the straight lines BC and BA will intersect each other during their traveling in some point that is always changing its position together with them. By this point a certain line such as BZH is described in the space between the straight lines BA and AD and the arc BED, concave in the same direction <as BED>, which appears to be useful, among other things, for finding a square equal to a given circle.[1]

And its principal *symptoma* is of the following sort. Whichever arbitrary <straight line> is drawn through in the interior toward the arc, such as AZE, the straight line BA will be to the <straight line> ZT as the whole arc <BED is> to the arc ED. For this is obvious from the *genesis* of the line.

Criticism of the Quadratrix Under the Description via Motions (Sporus)

#31 Sporus, however, is with good reason displeased with it, on account of the following <observations.>[2]

For, first of all, he[3] takes into the assumption the very thing for which it <i.e., the quadratrix> seems to be useful. For how is it possible when two points start from B, that they move, the one along the straight line to A, the other along the arc to D, and come to a halt <at their respective end points> at the same time, unless the ratio of the straight line AB to the arc BED is known beforehand? For the velocities of the motions must be in this ratio, also.[4] Also, how do they think that they[5] come to a halt simultaneously, when they use indeterminate velocities, except that it might happen sometime by chance; and how is that not absurd?

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[1] The quadratrix can be used also for the division of an angle in any given ratio (probably its original use), and for problems related to this construction. Cf. Props. 35–38.

[2] The passage taken from Sporus differs significantly from the mathematical expositions in *Coll.* IV. Note, e.g., the rhetorical questions and the polemical style. Co p. 88 replaces the name "Sporos" with the Latin word "spero." His paraphrase means: "I expect, however, that this line justifiedly and deservedly does not satisfy, for the following reasons." The replacement changes the meaning of the introductory sentence, and indeed of the whole passage criticizing the quadratrix considerably.

[3] The Greek text uses the third person singular. It is unclear whom Sporus' argument targeted.

[4] The use of the notion "velocity" is not quite precise here. However, it is clear what Sporos means, and his argument is valid. In order to synchronize the two motions as required, one must know π – or else use an approximation to stand in for it. However, π is exactly what the curve is supposed to exhibit in construction. Co p. 88 paraphrases "motuum velocitates." Hu 254, 7 emends A's elliptical "ἀναγκαῖον." For a parallel construction, without emendation, see, however 270, 11/12 Hu.

[5] The reading πῶς οἴονται (how do they think) as given in A, was kept. Both Hultsch and Treweek reject it in favor of the reading πῶς οἷόν τε (254, 8 Hu + app/ Tr. 109, 11), attested in the minor manuscripts. Co p. 88 paraphrases "quo pacto arbitrantur."

Furthermore, however, its endpoint, which they use for the squaring of the circle, i.e.: the point in which it intersects the straight line AD, is not found <by the above generation of the line>. Consider what is being said, however, with reference to the diagram set forth. For when the <straight lines> CB and BA, traveling, come to a halt simultaneously, they will <both> reach AD, and they will no longer produce an intersection in each other. For the intersecting stops when AD is reached,[1] and this <last> intersection would have taken place as the endpoint of the line,[2] the <point> where it meets the straight line AD. Except if someone were to say that he considers the line to be produced, as we assume straight lines <to be produced>, up to AD. This, however, does not follow from the underlying principles, but <one proceeds> just as if the point H were taken after the ratio of the arc to the straight line had been taken beforehand.[3] Without this ratio being given,[4] however, one must not,[5] trusting in the opinion of the men who invented the line, accept it,[6] since it is rather mechanical.[7] Much rather, however, one should accept the problem that is shown by means of it.[8]

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[1] Restoring A's reading πρὸς (when) instead of Hultsch's πρὸ (= before; cf. 254, 16 Hu app).

[2] Restoring, with Tr 109, 20, the reading of A.

[3] For an extension of the quadratrix to the base line one needs to know the direction. As the quadratrix does not have a constant direction, or even curvature, one needs, in the end, to know the position of H, and it would have to be determined beforehand, using the ratio of radius and circumference (π). My translation differs from Hultsch's Latin interpretation. Co has the following Latin paraphrase, rejected by Hultsch (p. 89 Co): Sed ut cumque sumatur punctum …, praecedere debet proportio circumferentiae ad rectam lineam.

[4] The Greek word (δοθῆναι) is the technical term from geometrical analysis. It is not certain (in fact perhaps unlikely) that Sporus, whom Pappus paraphrases here, intended it that way. What is certain, however, is that Pappus is going to interpret it in this strict technical sense for Props. 28 and 29. See below, and see the commentary on Props. 26–29.

[5] Accepting Hultsch's emendation οὐ for the difficult manuscript reading ἤ, kept in Tr 109, 26. Co p. 89 keeps the manuscript reading, and paraphrases as a question: Or should we… ? The disadvantage is that in that case one would have expected the question particle at the beginning of the sentence.

[6] I.e.: accept it as fully geometrical. The quadratrix itself (in the motion description) is not fully accepted; but note the upcoming remark on the mathematics *about* it. It is quite possible that Sporus and Pappus have different opinions on this matter. The issue cannot be pursued here.

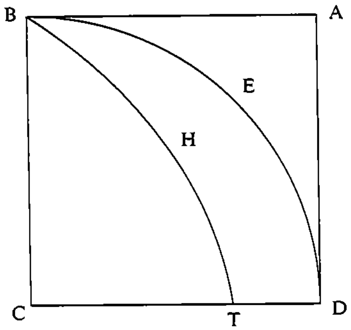
[7] Greek: μηχανικώτεραν. This word, used for the curve itself here, and not just for the way in which it is generated, is different from the label "ὀργανικῶς", i.e., "describable with an instrument". The latter was used in connection with Nicomedes' conchoid (cf. footnotes above). Hultsch deletes the phrase "and it is put to use by the students of mechanics for many problems" as an interpolation (254, 24–256, 1+app. Hu). There is indeed no evidence that the quadratrix played a major role in mathematical treatises on mechanics. A similar phrase occurs at 244, 20 Hu. See the introduction to Props. 19–30 in the commentary on "mechanical."

[8] Hultsch has changed the transmitted text considerably. His Latin paraphrase means: "but before I must report (assuming παραδοτέον) the problem that is solved on account of it." With Tr 110, 1–2, I keep the transmitted text. Co's paraphrase on p. 89 is compatible with this reading. See the commentary.

Prop. 26: Rectification of the Arc of a Quadrant

When a square ABCD is <given>, and the arc BED with center C,[1] and when the quadratrix BHT has come to be[2] in the above said way, it is shown that as the arc DEB <is> to the straight line BC, so <is> BC to the straight line CT.[3]

a

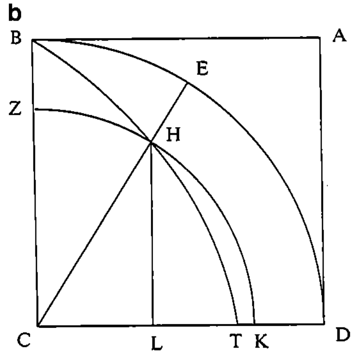


For:

If it is not <in that ratio to CT>, it will be <in that ratio> either to a <straight line> larger than CT or to one smaller.[4]

Assume first that, if this is possible, it is so to a larger <straight line> CK, and describe the arc ZHK with center C, intersecting the line in H, and <draw> HL as a perpendicular <onto CD>, and produce CH, after it has been joined, to E.

b



[1] Note the change of lettering in the diagram. Perhaps Prop. 26 was taken from a different source (Nicomedes, as opposed to Dinostratus, or else Sporus, for the curve's *genesis*?).

[2] Note that the quadratrix is posited at the outset. The upcoming argument will keep the problematical *genesis* of the curve out of sight, and use its *symptoma* only.

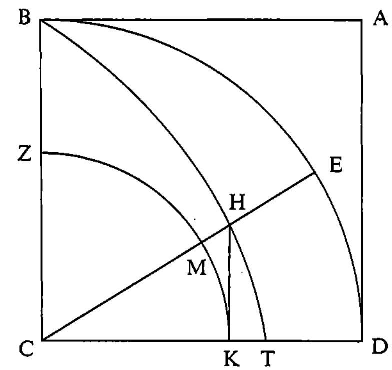
[3] This proportion will yield the construction of a straight line equal to arc DEB (Prop. 27).

[4] We get a classical proof via double reductio (so-called method of exhaustion). Apart from the (short and straightforward) alternative argument for the inverse of Prop. 13, this is the first, and the only, example for this argumentative technique in *Coll.* IV. On Prop. 26 see also Heath (1921, I, pp. 226–229).

Now, since as the arc DEB <is> to the straight line BC, so is BC, i.e.: CD, to CK,[1] whereas as CD <is> to CK, <so is> the arc BED to the arc ZHK (for as the diameter of a circle <is> to the diameter <of a second circle>, <so is> the circumference of the circle to the circumference <of the second circle[2]>), it is obvious that the arc ZHK is equal to the straight line BC.[3] And since, on account of the *symptoma* of the line, BC is to HL as the arc BED <is> to the arc ED, therefore, as the arc ZHK <is> to the arc HK, so <is> the straight line BC to HL,[4] also. And it has been shown that the arc ZHK is equal to the straight line BC. Therefore, the arc HK is equal to the straight line HL as well, which is absurd.[5] Therefore, it is not the case that as the arc BED <is> to the straight line BC, so is BC to a <straight line> larger than CT.

#32 I say, however, that it <i.e., BC> is not <in that ratio> to a <straight line> that is smaller, either.

c



[1] By assumption.

[2] This theorem is also used in Props. 36, 39, and 40, and a similar one in Prop. 30 (cf. notes ad locum). An explicit proof is given in *Coll.* V, 11 and *Coll.* VIII, 22. A possible justification might proceed as follows. XII, 2: circles have the ratio of the squares over their diameters; *Circ. mens.* I: circles have the ratio of the rectangles with radius and circumference as sides; V, 16 and VI, 1: circumferences have the ratio of diameters. V, 15: similar arcs have the ratio of diameters. The frequent occurrence of this motif may indicate that it is part of the special "jargon," a kind of basic tool within the "analytic track" of *symptoma*-mathematics of the third kind. Specifically, it might be a typical tool of Nicomedes. Nicomedes apparently systematically exploited properties of spiral lines, taking Archimedean arguments as a starting-point. Compare Pappus' remarks on the study of spiral lines and quadratrices as a central branch of geometry of the linear kind in the upcoming meta-theoretical passage.

[3] BC:CK = CD:CK = arc BED:BC (assumption); CD:CK = arc BED:arc ZHK ⇒ BC = arc ZHK (V, 9).

[4] arc BED:arc ED = BC:HL (*symptoma*). arc BED:arc ED = arc ZHK:arc HK (equal parts).

[5] arc ZHK:arc HK = BC:HL; arc ZHK = BC ⇒ arc HK = HL (V, 9). This is not possible, because 2HL is a chord under two times arc HK.

For if this is possible, assume that it is <in that ratio> to KC, and describe the arc ZMK with center C, and <draw> KH at right angles to CD intersecting the quadratrix in H, and produce CH, after it has been joined, to E. Similarly to what has been written above, then, we will show both that the arc ZMK is equal to the straight line BC, and that as the arc BED <is> to the <arc> ED, i.e.: <as> the <arc> ZMK <is> to the <arc> MK, so <is> the straight line BC to the <straight line> HK.[1] From these <observations> it is obvious that the arc MK will be equal to the straight line KH, which is absurd.[2] Therefore, it will not be the case that as the arc BED <is> to the straight line BC, so is BC to a <straight line> smaller than CT.

It has been shown, however, that it is not <in that ratio> to a larger one, either. Therefore, it <is in that ratio> to CT itself.

Prop. 27: Squaring the Circle

It is obvious, also, however, that when a straight line is taken as the third proportional to the straight lines TC and CB, it will be equal to the arc BED, and its fourfold to the circumference of the whole circle.[3] When, however, a straight line equal to the circumference of the circle has been found, it is very clear that it is rather easy indeed to put together a square equal to the circle itself. For the rectangle between the circumference of the circle and the radius is two times the circle, as Archimedes has shown.[4]

[1] Just as in the first part of the "exhaustion," one gets: CD:CK = arc BED:BC (assumption); arc BED:arc ZMK = CD:CK ⇒ arc ZMK = BC (V, 9). arc BED:arc ED (= arc ZMK:arc MK) = BC:HK (*symptoma*).

[2] HK must be larger than arc MK. I am not aware of an elementary geometrical argument in ancient geometry for this (correct) statement. Hultsch and Ver Eecke (1933b) ad locum refer to an argument that can be reconstructed from (Ps.-) Euclid, *Catoptrics* 8.

[3] Construct the third proportional s for TC and CB (VI, 11): TC:BC = BC:s; TC:BC = BC:arc BD (Prop. 26 with V, 16) ⇒ s = arc BD. Then 4 s is equal in length to the circumference of the circle.

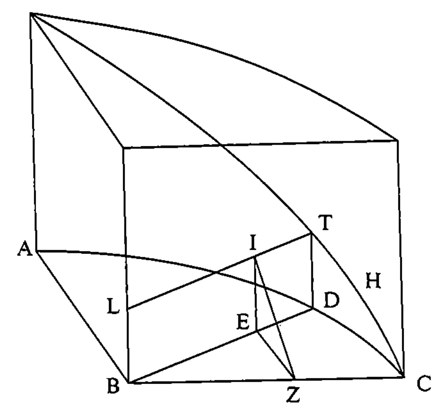
[4] *Circ mens.* I. This rectangle can be transformed into a square via II, 14.

Prop. 28: Analytical Determination of the Quadratrix from an Apollonian Helix

#33 Now, this *genesis* of the curve is, as has been said, rather mechanical[1]; it can, however, be made the subject of a geometrical analysis[2] by means of loci on surfaces in the following way.

<Let> the quadrant ABC of a circle <be *given*> in position, and assume that BD has been drawn through the interior arbitrarily, and also a perpendicular EZ onto BC, which has a *given* ratio to the arc DC.

<I claim> that E lies on a <uniquely determined> line.[3]



[1] Here Pappus picks up the discussion before Prop. 26, on the generation of the quadratrix via motions and the mathematical status of the quadratrix.

[2] ἀναλύεσθαι; since this is a technical term, clearly referring back to the technique of analysis (cf. Props. 4–12, and 31 ff.), Hultsch's Latin paraphrase "problema solvitur" does not capture the meaning and is in fact misleading. What is "analyzed" here is not the problem of squaring the circle, but the *genesis* of the quadratrix. Co paraphrases "lineae ortus … resolvi potest (p. 90). Both Prop. 28 and Prop. 29 provide a *resolutio* in the sense that they show that the quadratrix is *given*, if an Apollonian helix or an Archimedean spiral is posited (i.e., taken as *given*). See the commentary.

[3] With EZ: arc DC *given*, E will be shown to lie on a line that is determined relative to a certain helix, which is assumed as *given*. This characterization is independent from the *genesis* of the line via motions, which has been disqualified as conceptually inconsistent. It is not constructive, however, but rather a characterization via implicit relations. Note that the analysis is quite general in the sense that the ratio which is taken as *given* is not assumed to be the ratio of arc and radius, as in the quadratrix. Co p. 90, B is, in my view, mistaken when he assumes that. For each *given* ratio, the analysis shows that a unique line is determined by it via the intersection of the surface related to a *given* cylindrical helix and a *given* plane. For the special case of a ratio equal to arc ABC:AB, this line will be the quadratrix. Compare the end of Prop. 28, and Hultsch, * on p. 259 and #2 on p. 261.

For:

Consider the surface of a right cylinder over the arc ADC,[1] and described in it a helix CHT, *given* in position,[2] and <let> TD be the side of the cylinder,[3] and draw EI and BL at right angles to the plane of the circle, and finally, draw TL through T as a parallel to BD.[4] Since the ratio of the straight line EI to the arc DC is *given* on account of the helix,[5] whereas the ratio of EZ to the <arc> DC is *given*, also, the ratio of EZ to EI will be *given*, also.[6] And ZE and EI are (*given*) as parallels in position.[7] Therefore, the joining <straight line> ZI is <*given*> as parallel in position, also.[8] And it is a perpendicular onto BC. Therefore, ZI lies in a plane intersecting <the cylinder,>[9] so that I <lies there>, also. It <lies>, however, on a surface belonging to the cylinder as well[10] (for TL travels through both the helix THC and the straight line LB, which is also itself *given* in position, while it remains parallel throughout to the underlying plane). Therefore, I lies on a <uniquely determined> line,[11] so that E does so as well.[12]

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[1] Co p. 90 D assumes, mistakenly in my view, that the height of the quarter cylinder constructed has to be equal to AB, and that the defining ratio of linear upward motion to rotation is that of arc ADC:BC.

[2] For a definition of the helix cf. Heron, def. 8, 1 and 8, 2.

[3] TD is perpendicular to the plane of the circular quadrant; it is now considered as the height of the cylinder under discussion.

[4] We create a rectangle BDTL, with E on BD and I on HL, and EI parallel to DT.

[5] This ratio is implicit in the helix as the relation of rotation and upward motion in its *genesis*.

[6] *Data* 8. The sentence is truncated in A. Above, I have translated the text as emended by Hultsch (260, 8–10 + app. Hu, see also #3 on p. 261 Hu). Tr 111, 27–112, 3 prints an alternative reconstruction, closer to the actual manuscript reading, and therefore perhaps preferable (see the apparatus in the Greek text).

[7] *Data*, def. 15. When a line is *given* in position, the parallel to it through a *given* point is said to be *given as a parallel in position* (para thesei).

[8] *Data* 41 and 29.

[9] The manuscript is severely damaged by water in this place, and the text is not legible (cf. the apparatus in the Greek text). Hultsch's emendation "ἐν τέμνοντι ἄρα" leaves open the possibility that the intersecting plane is determined by EZ and ZI or else by BC and ZI, whereas Treweek's emendation identifies the plane in question as the one determined by EZ and ZI. The version with BC/ZI as intersecting plane has the drawback that the endpoint Z of the intersection line is not uniquely determined. The plane has BC in common with the garland – shaped surface created by the helix. The version with EZ /ZI has the drawback that one would have to know the exact position of either EZ or ZI, and it is unclear how that could be accomplished at this stage of the analysis. I therefore prefer the former version. See the commentary.

[10] Severe damage to the manuscript text here; cf. apparatus to the Greek text for different emendations suggested.

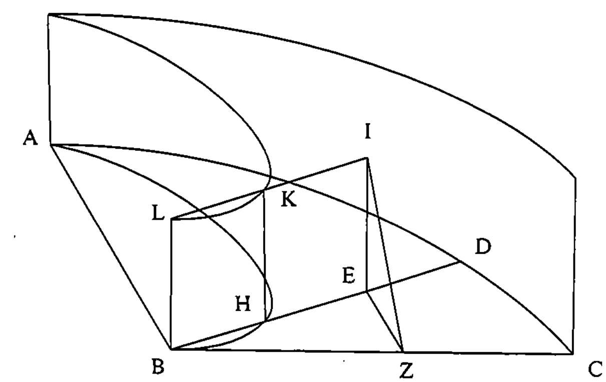
[11] The point I lies on the line of intersection between the abovementioned plane and the surface created by the ascending line BC in the cylinder.

[12] E lies on the projection of the line created on the cylindrical surface onto the plane.

Now, this has been subjected to analysis in a general way.[1] When, however, the ratio of the straight line EZ to the arc DC is the same as that of BA to the <arc> ADC, the above-mentioned line quadratrix comes to be.

Prop. 29: Analytical Determination of the Quadratrix from the Archimedean Spiral

#34 It can, however, also be made the subject of analysis[2] by means of a spiral described in the plane, in a similar way.



For:

Assume that the ratio of EZ to the arc DC is the same as the <ratio> of AB to the arc ADC,[3] and that in the time in which the straight line AB, moving around B passes through the arc ADC, a point on it, starting from A, arrives at B when AB

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[1] ἀνελύθη. Compare the introductory phrase of Prop. 28 with note.

[2] The Greek text, again, has ἀναλύεσθαι. Compare the introductory phrase of Prop. 28.

[3] We are starting from a configuration with a section of a circle ABC and a part of it DBC. The arc ADC is not necessarily the arc of a quadrant (Co p. 91 is probably mistaken in assuming so). An Archimedean spiral will be assumed in it, and the analysis will show that any such configuration with spiral will determine a unique quadratrix-type line, though not necessarily the quadratrix itself. When a spiral is chosen with an inbuilt ratio equal to the ratio of the circumference of a quadrant to the radius, we get the quadratrix.

takes the position of CB,[1] and that it creates the spiral BHA. Then the arc ADC is to the <arc> CD as AB is to BH,[2] and alternate <this equation.>[3]

But EZ <is in that ratio> to <arc> DC, also.[4] Therefore, BH is equal to ZE.[5] Draw KH at right angles to the plane, equal to BH. Then K lies in a cylindroid surface over the spiral.[6]

It <lies>, however, also on the surface of a <uniquely determined> cone (for BK, when it is joined, turns out to lie on the surface of a cone inclined at an angle of 45° toward the underlying <plane>, and drawn through the *given* <point> B <as vertex>[7]). Therefore, K <lies> on a <uniquely determined> line.[8]

Draw LKI through K as a parallel to EB, and BL and EI at right angles to the <underlying> plane.[9] Then LKI (lies) on a plectoid[10] surface (for it travels both through the straight line BL, which is *given* in position and through the line, *given* in position, on which K <lies>). Therefore, I lies on a <uniquely determined> surface, also. But it also lies on a <uniquely determined> plane (for ZE is equal to EI, since it is also equal to BH, and ZI turns out to be *given* as a parallel in position, since it is a perpendicular onto BC). Therefore, I <lies> on a <uniquely determined> line,[11] so that E, also, <lies on a uniquely determined line>.

And it is clear that, when the angle ABC is a right <angle>, the above-mentioned line "quadratrix" comes to be.

[1] Compare the description of the *genesis* of the spiral before Prop. 19. The direction of the travel through AB and through the circumference is reversed in comparison to the former version. Also, the spiral is inscribed not in a full circle, but in a sector. The above translation accepts Hultsch's emendations in 262, 7–9. Tr 112, 17/18 prints Hultsch' s version, but notes that one might have emended ΓΔA in 262, 7 Hu and kept the manuscript reading for the rest of the sentence. Then the spiral is generated exactly like the one in Prop. 19. In the Greek text, Tr's suggestion was implemented (cf. apparatus).

[2] *Symptoma* of the spiral, following directly from the *genesis*.

[3] AB:arc ADC = BH:arc DC.

[4] By assumption.

[5] V, 9 or V, 15.

[6] This surface is built up over the spiral as limiting line of the base. Co p. 91/92 assumes a different situation, with a full cylinder quadrant and an inscribed Apollonian helix, in addition to the cylindroid. For yet another reconstruction cf. Knorr (1986, p. 166 f).

[7] By construction, HK = BH, and $\angle BHK = \pi/2$. Therefore, $\angle HBK = \pi/4$.

[8] K lies on the line created by the intersection of the two surfaces mentioned, cylindroid over the spiral, and surface of the cone with vertex B.

[9] Without loss of generality, L and I can be chosen as the points of intersection between the parallel to BD through T and the straight lines EI, BL.

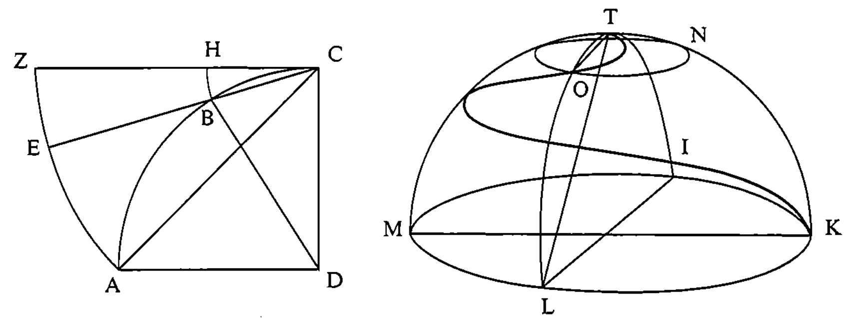
[10] The Greek word πλεκτοειδῆς (πληκτοειδῆς in A, Tr 112, 27, and Ver Eecke ad locum) is used here as a technical term the context for which is now lost. Following Hultsch, I have left it untranslated. What a plectoid surface looks like can be derived from the description given here by Pappus. There is no other, independent source.

[11] I lies on the line created by the intersection of the surfaces mentioned.

Prop. 30: *Symptoma*-Theorem on the Archimedean Spherical Spiral

Prop. 30: Surfaces Cut Off by a Spiral on a Hemisphere

#35 Just as a certain spiral is contemplated in the plane when a point travels along a straight line that describes a circle, and in solids when a point travels along one of its sides,[1] while it describes a certain surface, so it is in fact a natural next step[2] to contemplate a spiral described on a sphere, in the following way.[3]



Let KLM be a maximum circle in a sphere with point T as pole, and assume that starting from T the quadrant TNK of a maximum circle is described, and that the arc TNK, traveling around T, which remains in its position, along the surface <of the sphere>, in the direction of the parts <containing> L and M, comes to a halt again in the same position, whereas a certain point traveling on it, starting from T, arrives at K. Now, it describes a certain spiral, such as TOIK on the surface,[4] and

[1] Severe damage to the manuscript text; see the apparatus for different conjectures.

[2] The Greek text has ἀκολουθόν; once again, we have a context in which the word cannot signify a logical derivation, and must mean a next step in a somewhat orderly fashion. See the commentary on analysis-synthesis in the introduction to Props. 4–12.

[3] Although this introductory paragraph draws an explicit connection to Props. 19, 28, and 29, the path of reasoning about the spiral line is very different from Props. 28 and 29. It shows affinities to Prop. 21 ("meta-mechanical" path of reasoning about the motion curves, quasi-infinitesimals, limit process, no analysis).

[4] Compare the *genesis* of the plane spiral in Prop. 19. The ratio of the velocities for the two synchronized motions involved in Prop. 30 is simply 4:1. Cf. equations in polar coordinates: spherical spiral $\rho = 1/4\ \omega$, plane spiral in Prop. 19 $\rho = (1/2\pi)\omega$, plane spiral in *SL* $\rho = a\omega$, where a is a natural number or a ratio of two numbers. The spherical spiral by motions can be constructed in thought exactly.

whichever arc of a maximum circle is described starting from T,[1] it will have to the arc KL the ratio that the <arc> LT has to the <arc> TO.[2]

Now, I claim that, when the arc ABC of a quadrant of the maximum circle in the sphere with center D is set out, and CA is joined, the sector ABCD turns out to be to the segment ABC as the surface of the hemisphere <is> to the surface cut off <from it> between the spiral TOIK and the arc KNT.[3]

For:

Draw CZ as a tangent to the arc <ABC>, and describe the arc AEZ <of the circle> through A with center C. Then the sector ABCD is equal to the <sector> AEZC (for the angle at D is two times the angle ACZ, whereas the square over DA is half the square over AC[4]). Therefore, <we need to show> that, as the said surfaces are to each other, so <is> the sector AEZC to the segment ABC, also.[5]

Let the arc KL be a part of the whole circumference of the circle, and the <arc> ZE the same part of the <arc> ZA, and join EC. Now, the <arc> BC will be the same part of the <arc> ABC.[6] However, whichever part the <arc> KL is of the whole circumference, the <arc> TO is that same part of the <arc> TOL, also.[7] And the <arc> TOL is equal to the <arc> ABC. Therefore, the <arc> TO is equal to the <arc> BC as well.

Describe the circle ON through O with pole T, and the <arc> BH through B with center C. Now, since as the surface LKT on the sphere <is> to the <surface> OTN, so <is> the whole surface of the hemisphere to the surface of the section the spherical radius of which is TO,[8] whereas as the surface of the hemisphere

[1] Cf. the full circle going through LOTI, intersecting the spiral in O. Co p. 93, C is mistaken in assuming that arc KL is fixed as a quarter circle now. A division $1:2^n$ is likely (cf. Prop. 21).

[2] The *symptoma* of the spherical spiral is read off directly from the *genesis* via motions; cf. the plane spiral (Prop. 19) and the quadratrix (before Prop. 26), but contrast the conchoid (before Prop. 23). I have based the translation on Hultsch's emendations in 264, 16/17 Hu. Tr 113, 20–22 prints an emendation that is closer to the manuscript reading and is perhaps preferable (cf. apparatus).

[3] The formulation of the protasis is analogous to Prop. 21. An area theorem is expressed in terms of numerical ratios. Cf. Prop. 16: a theorem on a sequence of ratios of lines is expressed in numerical ratios.

[4] $\angle ADC = \angle ZCD = \pi/2$ (III, 18); $\angle ACZ = \angle ACD = \pi/4$ (ΔADC isosceles). $AC^2 = 2AD^2$ (I, 47). 2(sector AZC):sector ACD = $AC^2:AD^2 = 2AD^2:AD^2$ (XII, 2) = 2:1.

[5] The configuration investigated has been transformed to a situation of analogy between surface with surface "inside" and sector with segment "inside"; cf. Prop. 21's use of a parallel auxiliary configuration with rotation cylinders, and investigation via parallel processes of continuous inscription.

[6] arc ZE:arc ZA = $\angle ZCE:\angle ZCA$ (VI, 33); $\angle CDA = 2\angle ZCA$; $\angle CDB = 2\angle ZCE$ (III, 32 and III, 20) $\Rightarrow$ arc ZC:arc ZA = $\angle ZCE:\angle ZCA = \angle CDB:\angle CDA$ = arc CB:arc CA (VI, 33).

[7] *Symptoma* of the spiral.

[8] V, 15 (surface LKT:surface OTN = surface hemisphere:full surface ONT).

<is> to the surface of the section, so is the square over the straight line joining T and L to the square over the <straight line joining> T and O,[1] or the square over EC[2] to the square over BC, therefore as the sector KLT in the surface <is> to the <sector> OTN, so will the sector EZC be to the <sector> BHC.[3] Similarly we will show that, also, as all the sections in the hemisphere that are equal to KLT, taken together (they are <when put together> the whole of the surface of the hemisphere), <are> to the sections described around the spiral that are of the same order as OTN, taken together, so <are> all the sectors in AZC that are equal to EZC, taken together, i.e.: <so is> the whole sector AZC, to the <sectors> circumscribed around the segment ABC that are of the same order as <the sector> CBH, taken together.

In the same way it will also be shown, however, that as the surface of the hemisphere <is> to the sections inscribed in the spiral, so <is> the sector AZC to the sectors inscribed in the segment ABC, so that as the surface of the hemisphere <is> to the surface cut off by the spiral, so <is> the sector AZC, i.e.: the sector ABCD, to the segment ABC.[4]

Addition:

On account of this result one gathers, however, that the surface cut off between the spiral and the arc TNK is eight times the segment ABC (since the surface of the hemisphere <is eight times> the sector ABCD, also[5]), whereas the surface (cut off) between the spiral and the base of the hemisphere is eight times the triangle ACD, i.e.: <it is> equal to the square over the diameter of the sphere.[6]

[1] Surface hemisphere = 2 maximum circle (*Sph. et Cyl.* I, 33); circle with radius TL:maximum circle = TL^2:(radius hemisphere)^2 (XII, 2) = 2:1 (I, 47); $\Rightarrow$ surface of hemisphere = circle with radius TL; Surface of sphere through O, N with pole T = circle with radius TO (*Sph. et Cyl.* I, 42:); $\Rightarrow$ Surface hemisphere:surface ONT = circle TL:circle TO = $TL^2:TO^2$; cf. Co p. 94, K for a slightly different path of reasoning.

[2] By construction, TL = AC = ZC, and BC = TO as chords under equal arcs (III, 29).

[3] XII, 2; V, 15 (circles have ratio of squares over radii); the same proportion holds for equal parts. Ver Eecke (1933b, p. 204, #4) refers to *Sph. et Cyl.* I, 42/43 here.

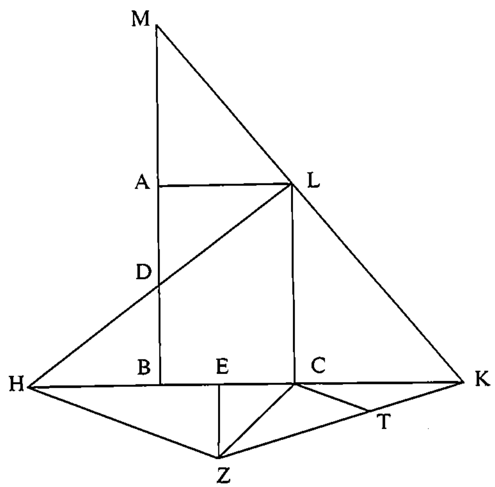
[4] An implicit limit process is used (cf. Prop. 21). The sought areas are analogously enclosed between all circumscribed and all inscribed composite circular areas/spherical sections. By choosing the arcs involved in the construction ever smaller, the desired lines and areas are approximated.

[5] *Sph. et Cyl.* I, 33 (surface of the complete sphere = 4 area of maximum circle). *Sph. et Cyl.* I, 35: surface hemisphere = 8 quadrants of maximum circle. Thus, surface above spiral = 8 segments.

[6] We compare the remainders after subtraction. Since surface above spiral = 8 segments, we get that surface hemisphere – surface above spiral = surface below spiral = $8\Delta ACD$. $8\Delta ACD = 8(1/2\ AD^2) = 4AD^2 = (2AD)^2$.

ὡς ἄρα ἡ ΜΑ πρὸς ΑΒ, οὕτως ἡ ΒΓ[1] πρὸς ΓΚ. καὶ ἔστι τῆς μὲν ΑΒ ἡμίσεια ἡ ΑΔ, τῆς δὲ ΒΓ διπλῆ ἡ ΓΗ· ἔσται ἄρα καὶ ὡς ἡ ΜΑ πρὸς ΑΔ, οὕτως ἡ ΗΓ πρὸς ΚΓ. ἀλλ' ὡς ἡ ΗΓ πρὸς ΓΚ, οὕτως ἡ ΖΘ πρὸς ΘΚ διὰ τὰς παραλλήλους < τὰς > ΗΖ ΓΘ[2]· καὶ συνθέντι ἄρα ὡς ἡ ΜΔ πρὸς ΔΑ, ἡ ΖΚ πρὸς ΚΘ. ἴση δὲ ὑπόκειται καὶ ἡ ΑΔ

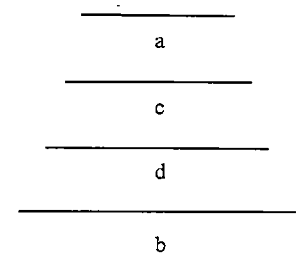
f. 47 (Props. 24 and 25, and Quadratrix)
τῇ ΘΚ, ἐπεὶ καὶ τῇ ΓΖ ἴση ἐστὶν ἡ ΑΔ[3]· ἴση ἄρα καὶ ἡ ΜΔ τῇ ΖΚ· ἴσον ἄρα καὶ τὸ ἀπὸ ΜΔ τῷ ἀπὸ ΖΚ. καὶ ἔστι τῷ μὲν ἀπὸ ΜΔ ἴσον τὸ ὑπὸ ΒΜΑ[4] μετὰ τοῦ ἀπὸ ΔΑ, τῷ δὲ ἀπὸ ΖΚ ἴσον ἐδείχθη τὸ ὑπὸ ΒΚΓ μετὰ τοῦ ἀπὸ ΖΓ, ὧν τὸ ἀπὸ ΑΔ ἴσον τῷ ἀπὸ ΓΖ. ἴση γὰρ ὑπόκειται ἡ ΑΔ τῇ ΓΖ. ἴσον ἄρα καὶ τὸ ὑπὸ ΒΜΑ τῷ ὑπὸ ΒΚΓ· ὡς ἄρα ἡ ΜΒ πρὸς ΒΚ, ἡ ΓΚ πρὸς ΜΑ[5]. ἀλλ' ὡς ἡ ΒΜ πρὸς ΒΚ, ἡ ΛΓ πρὸς ΓΚ· ὡς ἄρα ἡ ΛΓ πρὸς ΓΚ, < ἡ ΓΚ πρὸς ΑΜ. ἔστι δὲ καὶ ὡς ἡ ΜΒ πρὸς ΒΚ, >[6] ἡ ΜΑ πρὸς ΑΛ· καὶ ὡς ἄρα ἡ ΛΓ πρὸς ΓΚ, ἡ ΓΚ πρὸς ΑΜ, καὶ ἡ ΑΜ πρὸς ΑΛ.



Prop. 25
<u>#29</u> τούτου δειχθέντος πρόδηλον ὅπως δεῖ κύβου δοθέντος κύβον ἄλλον εὑρεῖν κατὰ τὸν δοθέντα λόγον. ἔστω γὰρ ὁ

[1]*ΒΓ* A Hu *ΒΚ* Tr
[2]*ἡ ΖΓΘ* A corr. Hu, Tr *τὰς ΗΖ ΓΘ* Eut.
[3]*ἐπεὶ καὶ τῇ ΓΖ ἴση ἐστὶν ἡ ΑΔ* del. Hu
[4]*τὸ ἀπὸ ΒΜΑ* ABS corr. Hu, Tr *τὸ ὑπὸ ΒΜΑ* Eut.
[5]*ἡ ΛΓ πρὸς ΓΚ* ABS corr. Hu, Tr *ἡ ΚΓ πρὸς ΑΜ* Eut.
[6]*ἡ ΓΚ...πρὸς ΒΚ* om. ABS restit. Hu, Tr ἡ ΓΚ *πρὸς ΑΜ. ἔστι δὲ καὶ ὡς ἡ ΛΓ πρὸς ΓΚ* Eut.

δοθεὶς λόγος τῆς Α εὐθείας πρὸς τὴν Β, καὶ τῶν *ΑΒ*[1] δύο μέσαι ἀνάλογον κατὰ τὸ συνεχὲς εἰλήφθωσαν αἱ ΓΔ[2]· ἔσται ἄρα ὡς ἡ Α πρὸς τὴν Β, οὕτως ὁ ἀπὸ τῆς Α κύβος πρὸς τὸν ἀπὸ τῆς Γ κύβον, τοῦτο γὰρ δῆλον ἐκ τῶν στοιχείων.



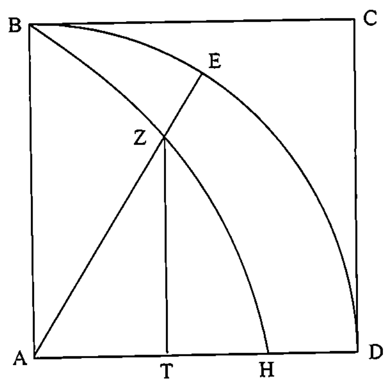
Quadratrix
<u># 30</u> εἰς τὸν τετραγωνισμὸν τοῦ κύκλου παρελήφθη τις ὑπὸ Δεινοστράτου καὶ Νικομήδους[3] γραμμὴ καί τινων ἄλλων νεωτέρων ἀπὸ τοῦ περὶ αὐτὴν συμπτώματος λαβοῦσα τοὔνομα, καλεῖται γὰρ ὑπ' αὐτῶν τετραγωνίζουσα καὶ γένεσιν ἔχει τοιαύτην. ἐκκείσθω τετράγωνον τὸ *ΑΒ ΓΔ*[4], < καὶ >[5] περὶ κέντρον τὸ Α περιφέρεια γεγράφθω < ἡ >[6] ΒΕΔ, καὶ κινείσθω ἡ μὲν ΑΒ οὕτως ὥστε τὸ μὲν Α σημεῖον μένειν, τὸ δὲ Β φέρεσθαι κατὰ τὴν ΒΕΔ περιφέρειαν, ἡ δὲ ΒΓ παράλληλος ἀεὶ διαμένουσα τῇ ΑΔ τῷ Β σημείῳ φερομένῳ[7] κατὰ τῆς ΒΑ συνακολουθείτω[8], < καὶ >[9] ἐν ἴσῳ χρόνῳ ἥτε ΑΒ κινουμένη[10] ὁμαλῶς τὴν ὑπὸ ΒΑΔ γωνίαν, τουτέστιν

[1]distinx. B Hu, Tr
[2]distinx. BS Hu, Tr
[3]*ὑπὸ δεινοστράτου καὶ νικοδήμου* AB3 corr. S Hu, Tr *ὑπὸ νικοστράτου* B *νικομήδου* B1 To
[4]coniunx. B Hu, Tr
[5]*καὶ* add. To, Hu, Tr
[6]*ἡ* add. Hu, Tr
[7]*τῷ* B *σημεῖον φερον ἐν ᾧ* ABS corr. To, Hu, Tr
[8]*κατὰ τῆς* B *συνακολουθεῖ τῷ* AS corr. To, Hu, Tr *συνακολουθεῖ τὸ* B
[9]*καὶ* add. To, Hu, Tr
[10]*κινουμένης* AB3S *κινουμένη* B1 Hu, Tr

τὸ Β σημεῖον τὴν ΒΕΔ <περιφέρειαν>[1], διανυέτω, καὶ ἡ ΒΓ τὴν ΒΑ εὐθεῖαν παροδευέτω, τουτέστιν τὸ Β σημεῖον κατὰ τῆς ΒΑ φερέσθω. συμ-

f. 47v (Quadratrix and Sporos)

βήσεται δῆλον[2] τῇ ΑΔ εὐθείᾳ ἅμα ἐφαρμόζειν ἑκατέραν[3] τήν τε ΑΒ καὶ τὴν ΒΓ. τοιαύτης δὴ γινομένης κινήσεως τεμοῦσιν ἀλλήλας ἐν τῇ φορᾷ αἱ ΒΓ ΒΑ εὐθεῖαι κατά τι σημεῖον αἰεὶ συμμεθιστάμενον αὐταῖς, ὑφ' οὗ σημείου γράφεταί τις ἐν τῷ μεταξὺ τόπῳ τῶν τε ΒΑΔ εὐθειῶν καὶ τῆς ΒΕΔ περιφερείας γραμμὴ ἐπὶ τὰ αὐτὰ κοίλη, οἵα ἐστὶν ἡ ΒΖΗ, <ἣ>[4] καὶ χρειώδης[5] εἶναι δοκεῖ πρὸς τὸ τῷ δοθέντι κύκλῳ τετράγωνον ἴσον εὑρεῖν. τὸ δὲ ἀρχικὸν αὐτῆς σύμπτωμα τοιοῦτόν ἐστιν· ἥτις γὰρ ἂν διαχθῇ τυχοῦσα <εὐθεῖα πρὸς τὴν περιφέρειαν, ὡς ἡ ΑΖΕ, ἔσται ὡς *ἡ ὅλη*>[6] περιφέρεια πρὸς τὴν ΕΔ, ἡ ΒΑ [*περιφέρεια*][7] εὐθεῖα πρὸς τὴν ΖΘ· τοῦτο γὰρ ἐκ τῆς γενέσεως τῆς γραμμῆς φανερόν ἐστιν.



#31 δυσαρεστεῖται δὲ αὐτῇ[8] ὁ Σπόρος[9] εὐλόγως διὰ ταῦτα. πρῶτον μὲν γὰρ

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[1] *περιφέρειαν* add. To, Hu

[2] *δῆλον* A Hu *δηλονότι* vel *δὴ* coni. Hu *δὴ[λον]* Tr *[δῆλον]* Eberhard

[3] *ἑκατερα* A *ἑκατέρα* B corr. S Hu, Tr

[4] *ἣ* add. Hu, Tr

[5] *χρειῶδες* ABS corr. To, Hu, Tr

[6] *εὐθεῖα πρὸς τὴν περιφέρειαν, ὡς ἡ* ΑΖΕ, *ἔσται ὡς ἡ ὅλη* add. Tr *πρὸς τὴν περιφέρειαν, ὡς ἡ* ΑΖΕ, *ἔσται ὡς ὅλη ἡ* add. Hu *εὐθεῖα πρὸς τὴν περιφέρειαν, ὡς ἡ* ΒΖΕ, *ἔσται ὅλη ἡ ΒΕΔ* add. To

[7] *περιφέρεια* del. S Hu, Tr

[8] *αὐτῷ* coni. Hu

[9] *σπόρος* A corr. Hu, Tr

πρὸς ὃ δοκεῖ χρειώδης εἶναι πρᾶγμα, τοῦτ' ἐν ὑποθέσει[1] λαμβάνει. πῶς γὰρ δυνατόν, δύο σημείων ἀρξαμένων ἀπὸ τοῦ Β κινεῖσθαι, τὸ μὲν κατ' εὐθείας ἐπὶ τὸ Α, τὸ δὲ κατὰ περιφερείας ἐπὶ τὸ Δ ἐν[2] ἴσῳ χρόνῳ *συναποκαταστῆναι*[3] μὴ πρότερον τὸν λόγον[4] τῆς ΑΒ εὐθείας πρὸς τὴν ΒΕΔ περιφέρειαν ἐπιστάμενον; ἐν γὰρ τούτῳ τῷ λόγῳ[5] καὶ τὰ τάχη τῶν κινήσεων *ἀναγκαῖον*[6]. ἐπεὶ πῶς *οἴονται*[7] συναποκαταστῆναι[8] τάχεσιν ἀκρίτοις χρώμενα[9], πλὴν εἰ μὴ ἂν[10] κατὰ τύχην ποτὲ[11] συμβῇ[12]; τοῦτο δὲ πῶς οὐκ ἄλογον; ἔπειτα δὲ τὸ πέρας αὐτῆς ᾧ χρῶνται πρὸς τὸν τετραγωνισμὸν τοῦ κύκλου, τουτέστιν καθ' ὃ τέμνει σημεῖον τὴν ΑΔ εὐθεῖαν, οὐχ εὑρίσκεται. νοείσθω δὲ ἐπὶ τῆς προκειμένης τὰ λεγόμενα καταγραφῆς· ὁπόταν <γὰρ>[13] αἱ ΓΒ ΒΑ φερόμεναι συναποκατασταθῶσιν, ἐφαρμόσουσιν *τὴν* ΑΔ[14] καὶ τομὴν οὐκέτι ποιήσουσιν ἐν ἀλλήλαις· παύεται γὰρ ἡ τομὴ πρὸς *τῆς*[15] ἐπὶ τὴν ΑΔ ἐφαρμογῆς, ἥπερ τομὴ πέρας *ἂν*[16] ἐγένετο τῆς γραμμῆς καθ' ὃ τῇ ΑΔ εὐθείᾳ συνέπιπτεν. πλὴν εἰ μὴ λέγοι τις ἐπινοεῖσθαι προσεκβαλλομένην τὴν γραμμὴν ὡς ὑποτιθέμεθα τὰς εὐθείας ἕως τῆς ΑΔ· τοῦτο δ' οὐχ ἕπεται

f. 48 (Sporos and Prop. 26)

ταῖς ὑποκειμέναις ἀρχαῖς, ἀλλ' ὡς *ϑ* ἂν[17] ληφθείη τὸ Η σημεῖον προειλημμένου τοῦ τῆς περιφερείας πρὸς τὴν εὐθεῖαν λόγου. χωρὶς

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[1] *ὑ - ποθέσει* A

[2] *τὸ* ΔΕ Κ A corr. S Hu, Tr *τὸ δεη* B

[3] *συναποκαταστῆσαι* Hu

[4] *τολον* A *ὸν* superscriptum prima manu *τὸ ὅλον* B To *τὸν λόγον* S

[5] *ἐν γὰρ τῷ αὐτῷ λόγῳ* coni. Hu

[6] *ἀναγκαῖον* ABS Tr *ἀναγκαῖον εἶναι* (omisso posthac *ἐπεὶ*) To *ἀνάγκη εἶναι* Hu

[7] *ἐπει πως οιονται* (sine acc.) A *πῶς οἴονται γὰρ* To *ἐπεὶ πῶς οἷόν τε* BS Hu, Tr quo pacto arbitrantur Co

[8] *συναποκαταστῆσαι* coni. Hu

[9] *χρώμενον* coni. Hu

[10] *ἂν* del. To, probat et *συμβαίη* coni. Hu

[11] *τότε* A corr. To, Hu, Tr

[12] *συμβη* sine acc. A

[13] *γὰρ* add. Hu

[14] *τῇ ΑΔ* Hu, Tr *ἐπὶ τὴν ΑΔ* To

[15] *πρὸς τῆς* ABS *πρὸ τῆς* To, Hu, Tr

[16] *αὖ* Hu

[17] *ἀλλ' ὡς ϑ ἂν* AB To, Tr *ἄλλως ϑ ἂν* S *ἀλλ' ὡς ἂν* Hu

δὲ τοῦ δοθῆναι τὸν λόγον τοῦτον, οὐ[1] χρὴ τῇ[2] τῶν εὑρόντων ἀνδρῶν δόξῃ[3] πιστεύοντας παραδέχεσθαι τὴν γραμμὴν μηχανικωτέραν πῶς οὖσαν [καὶ εἰς πολλὰ προβλήματα χρησιμεύουσαν τοῖς μηχανικοῖς][4], *πολὺ* πρότερον παραδεκτέον ἐστὶ[5] τὸ δι' αὐτῆς δεικνύμενον πρόβλημα.

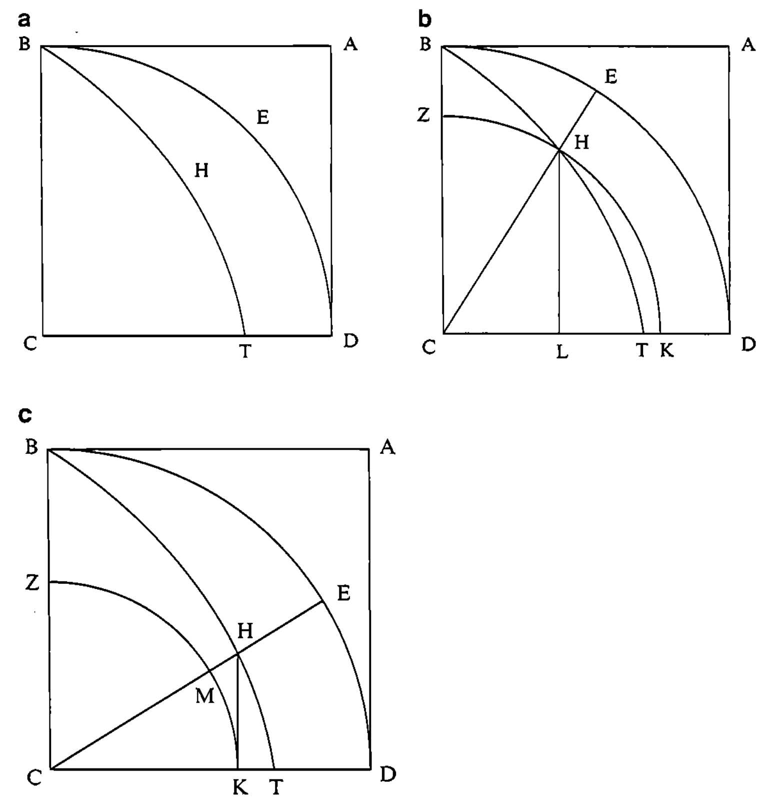
Prop. 26
τετραγώνου γὰρ ὄντος τοῦ ΑΒΓΔ[6] καὶ τῆς μὲν περὶ τὸ κέντρον τὸ Γ περιφερείας τῆς ΒΕΔ, τῆς δὲ ΒΗΘ[7] τετραγωνιζούσης γινομένης, ὡς προείρηται, δείκνυται, ὡς ἡ ΔΕΒ περιφέρεια πρὸς τὴν ΒΓ εὐθεῖαν, οὕτως ἡ ΒΓ πρὸς τὴν ΓΘ εὐθεῖαν. εἰ γὰρ *μή ἐστιν*[8], ἤτοι πρὸς μείζονα ἔσται τῆς ΓΘ ἢ[9] πρὸς ἐλάσσονα. ἔστω πρότερον, εἰ δυνατόν, πρὸς μείζονα τὴν ΓΚ, καὶ περὶ κέντρον τὸ Γ περιφέρεια ἡ ΖΗΚ γεγράφθω τέμνουσα τὴν γραμμὴν κατὰ τὸ Η, καὶ κάθετος ἡ ΗΛ, καὶ ἐπιζευχθεῖσα ἡ ΓΗ ἐκβεβλήσθω ἐπὶ τὸ Ε. ἐπεὶ οὖν ἐστιν ὡς ἡ ΔΕΒ περιφέρεια πρὸς τὴν ΒΓ εὐθεῖαν, οὕτως ἡ ΒΓ, τουτέστιν ἡ ΓΔ, πρὸς τὴν ΓΚ, ὡς δὲ ἡ ΓΔ πρὸς τὴν ΓΚ, ἡ ΒΕΔ περιφέρεια πρὸς τὴν ΖΗΚ περιφέρειαν, ὡς γὰρ ἡ διάμετρος τοῦ κύκλου πρὸς τὴν διάμετρον, ἡ περιφέρεια τοῦ κύκλου πρὸς τὴν περιφέρειαν, φανερὸν ὅτι ἴση ἐστὶν ἡ ΖΗΚ περιφέρεια τῇ ΒΓ ευθείᾳ. καὶ ἐπειδὴ διὰ τὸ σύμπτωμα τῆς γραμμῆς ἐστιν ὡς ἡ ΒΕΔ περιφέρεια πρὸς τὴν ΕΔ, οὕτως ἡ ΒΓ πρὸς τὴν ΗΛ, καὶ ὡς ἄρα ἡ ΖΗΚ πρὸς τὴν ΗΚ περιφέρειαν, οὕτως ἡ ΒΓ εὐθεῖα πρὸς τὴν ΗΛ. καὶ ἐδείχθη ἴση ἡ ΖΗΚ περιφέρεια τῇ ΒΓ εὐθείᾳ· ἴση ἄρα καὶ ἡ ΗΚ περιφέρεια τῇ ΗΛ εὐθείᾳ, ὅπερ ἄτοπον. οὐκ ἄρα ἐστὶν ὡς ἡ ΒΕΔ περιφέρεια πρὸς τὴν ΒΓ εὐθεῖαν, οὕτως ἡ ΒΓ πρὸς μείζονα τῆς ΓΘ.

<u>#32</u> λέγω δὲ ὅτι οὐδὲ πρὸς ἐλάσσονα. εἰ γὰρ δυνατόν, ἔστω πρὸς τὴν ΚΓ, καὶ περὶ κέντρον τὸ Γ περιφέρεια γεγράφθω ἡ ΖΜΚ, καὶ πρὸς ὀρθὰς τῇ ΓΔ ἡ ΚΗ τέμνου-

[1] *η* A *ἢ* BS To, Tr οὐ Hu
[2] *τη* A corr. Hu, Tr
[3] *δόξη* A corr. Hu, Tr
[4] *καὶ εἰς... μηχανικοῖς* del. Hu, Tr
[5] *πολὺ πρότερον παραδεκτέον ἐστὶ* ABS Tr *ἀλλὰ πρότερον παραδεκτέον ἐστὶ* Hu *παραδοτέον* coni. Hu cf. versio Latina
[6] *ΑΒΓ* AB1 corr. B2S Co, Hu, Tr
[7] *ΒΕΘ* A corr. To, Tr, Hu
[8] *μη εστιν* sine acc. A *μή ἐστιν* Tr *μὴ ἔστιν* Hu
[9] *τῆς ΓΘΗ* AB1corr. B2S Hu, Tr

σα τὴν τετραγωνίζουσαν κατὰ τὸ Η, καὶ ἐπιζευχθεῖσα ἡ ΓΗ

f. 48v (Prop. 26, 27, and 28)
ἐκβεβλήσθω ἐπὶ τὸ Ε. ὁμοίως δὲ τοῖς προγεγραμμένοις δείξομεν καὶ τὴν ΖΜΚ περιφέρειαν τῇ ΒΓ εὐθείᾳ ἴσην, καὶ ὡς τὴν ΒΕΔ περιφέρειαν πρὸς τὴν ΕΔ, τουτέστιν[1] ὡς τὴν ΖΜΚ πρὸς τὴν ΜΚ, οὕτως τὴν ΒΓ εὐθεῖαν [*πρὸς τὴν ΜΚ οὕτως τὴν ΒΓ εὐθεῖαν*][2] πρὸς τὴν ΗΚ. ἐξ ὧν φανερὸν ὅτι ἴση ἔσται ἡ ΜΚ περιφέρεια τῇ ΚΗ εὐθείᾳ, ὅπερ ἄτοπον. οὐκ ἄρα ἔσται ὡς ἡ ΒΕΔ περιφέρεια πρὸς τὴν ΒΓ εὐθεῖαν, οὕτως ἡ ΒΓ πρὸς ἐλάσσονα τῆς ΓΘ. ἐδείχθη δὲ ὅτι οὐδὲ πρὸς μείζονα· πρὸς αὐτὴν ἄρα τὴν ΓΘ.



[1] *τοῦτό ἐστιν* AB *τουτέστιν* S Hu, Tr
[2] bis scripta del. S Hu, Tr

Prop. 27

ἔστι δὲ καὶ τοῦτο φανερόν, ὅτι ἡ τῶν ΘΓ ΓΒ εὐθειῶν τρίτη ἀνάλογον λαμβανομένη[1] εὐθεῖα ἴση ἔσται τῇ ΒΕΔ περιφερείᾳ, καὶ ἡ τετραπλασίων αὐτῆς τῇ τοῦ ὅλου κύκλου περιφερείᾳ. εὑρημένης δὲ τῇ τοῦ κύκλου περιφερείᾳ ἴσης εὐθείας, πρόδηλον ὡς *δεῖ*[2] καὶ αὐτῷ τῷ κύκλῳ ῥᾴδιον ἴσον τετράγωνον συστήσασθαι· τὸ γὰρ ὑπὸ τῆς περιμέτρου τοῦ κύκλου καὶ τῆς ἐκ τοῦ κέντρου διπλάσιόν ἐστι τοῦ κύκλου, ὡς' Ἀρχιμήδης[3] ἀπέδειξεν.

Prop. 28

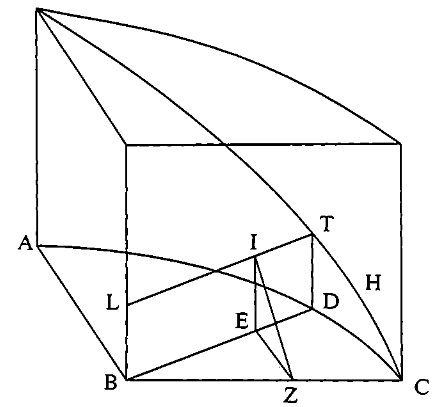
33 αὕτη μὲν οὖν ἡ γένεσις τῆς γραμμῆς ἐστιν, ὡς εἴρηται, μηχανικωτέρα· γεωμετρικῶς δὲ διὰ τῶν πρὸς ἐπιφανείαις τόπων ἀναλύεσθαι δύναται τὸν τρόπον τοῦτον. θέσει κύκλου τεταρτημόριον τὸ ΑΒΓ, καὶ διήχθω, ὡς ἔτυχεν[4], ἡ ΒΔ, καὶ κάθετος ἐπὶ τὴν ΒΓ ἡ ΕΖ λόγον ἔχουσα δοθέντα πρὸς τὴν ΔΓ περιφέρειαν· ὅτι πρὸς *γραμμὴν*[5] τὸ Ε. νοείσθω γὰρ ἀπὸ τῆς ΑΔΓ περιφερείας ὀρθοῦ κυλίνδρου ἐπιφάνεια, καὶ ἐν αὐτῇ ἕλιξ γεγραμμένη δεδομένη[6] τῇ θέσει ἡ ΓΗΘ, καὶ πλευρὰ[7] τοῦ κυλίνδρου ἡ ΘΔ, καὶ τῷ τοῦ κύκλου ἐπιπέδῳ ὀρθαὶ ἤχθωσαν αἱ *ΕΙΒΛ*[8] ἀνεσταμέναι ὀρθαί[9], διὰ δὲ τοῦ Θ[10] τῇ ΒΔ παράλληλος ἡ ΘΛ. ἐπεὶ <*δοθεὶς μέν ἐστι* >λόγος[11] τῆς *ΕΖ* εὐθείας[12] πρὸς τὴν ΔΓ περιφέρειαν, *τῆς δὲ ΔΓ*[13] διὰ τὴν ἕλικα λόγος πρὸς τὴν ΔΘ[14], ἔσται καὶ τῆς ΕΖ πρὸς ΕΙ[15] λόγος δοθείς. καὶ εἰσὶν αἱ ΖΕ ΕΙ παρὰ θέσει· καὶ ἡ ΖΙ ἄρα ἐπιζευ-

[1] *τρίτῃ ἀνάλογον λαμβανομένῃ* ABS corr. To, Hu, Tr
[2] *δεῖ* AB To *δὴ* S Hu, Tr
[3] *ἀρχιμήδης* A corr. Hu, Tr
[4] *ὡς ἐτύχην* AB1 corr. B3S Hu, Tr
[5] *πρὸς γραμμὴν* ABS *πρὸς γραμμῇ* Hu, Tr
[6] *ἕλιξ γεγραμμένη δεδομένη* A corr. B Hu, Tr
[7] *ΠΛ* AB corr. S Hu, Tr
[8] *ΕΙ ΒΛ* A *ΕΙ ΛΒ* B Hu, Tr
[9] *ἀνεσταμέναι ορθαί* del. Hu
[10] *διὰ δὲ τοῦ* K ABS corr. Co, Hu, Tr
[11] *ἐπι λόγος* ABS *ἐπίλογος* To *ἐπεὶ λόγος* Hu *δοθεὶς μέν ἐστι* add. Tr
[12] *τῆς* EI *ευθείας* Hu
[13] *περιφέρειαν, τῆς ΔΕ ΔΓ* ABS corr. Tr *περιφέρειάν ἐστιν δοθεὶς* Hu
[14] *τῆς δὲ ΔΓ διὰ τὴν ἕλικα λόγος πρὸς τὴν ΔΘ* (ABS) Tr *δοθεὶς δὲ καὶ ὁ τῆς ΕΖ λόγος πρὸς τὴν ΔΓ* Hu
[15] *Η* ABS corr. Tr

χθεῖσα παρὰ θέσει. καὶ ἔστιν κάθετος ἐπὶ τὴν ΒΓ· *ἐν ὀρθῷ*[1] ἄρα ἐπιπέδῳ ἡ ΖΙ, ὥστε καὶ τὸ Ι. ἔστιν δὲ καὶ ἐν *κυλινδροειδεῖ* ἐπι-

f. 49 (Props. 28, 29, and 30)

φανείᾳ[2], φέρεται γὰρ ἡ ΘΛ διά τε[3] τῆς ΘΗΓ ἕλικος καὶ τῆς ΛΒ εὐθείας καὶ αὐτῆς τῇ θέσει δεδομένης αἰεὶ παράλληλος οὖσα τῷ ὑποκειμένῳ ἐπιπέδῳ· πρὸς *γραμμὴν*[4] ἄρα τὸ Ι, ὥστε καὶ τὸ Ε. τοῦτο μέν οὖν ἀνελύθη καθόλου, ἂν δι ὁ τῆς ΕΖ εὐθείας πρὸς τὴν ΔΓ περιφέρειαν[5] λόγος ὁ αὐτὸς ᾖ[6] τῷ τῆς ΒΑ πρὸς τὴν *ΑΔ Γ*[7], ἡ προειρημένη τετραγωνίζουσα γίνεται γραμμή.

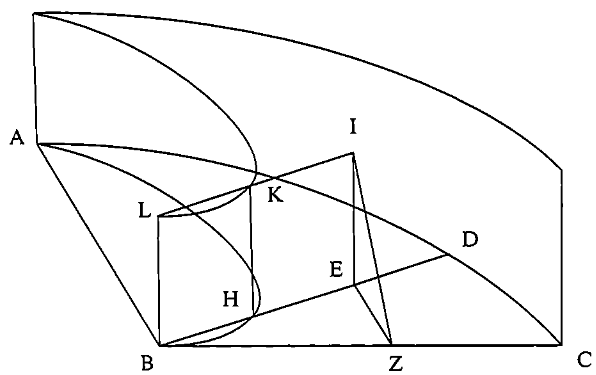


Prop. 29

34 δύναται δὲ καὶ διὰ τῆς ἐν ἐπιπέδῳ γραφομένης ἕλικος ἀναλύεσθαι τὸν ὅμοιον τρόπον. ἔστω γὰρ ὁ τῆς ΕΖ πρὸς τὴν ΔΓ < περιφέρειαν >[8] λόγος ὁ αὐτὸς τῷ τῆς ΑΒ πρὸς τὴν ΑΔΓ περιφέρειαν, καὶ ἐν ᾧ ἡ ΑΒ εὐθεῖα περὶ τὸ Β κινουμένη[9]

[1] *ὀρθῷ* (vix legibile) A Tr *τεμνόντι* Hu
[2] *ἐν κ////// ///φανείᾳ* A.........*ἐπιφανείᾳ* BS *ἐν κυλινδροειδεῖ ἐπιφανείᾳ* To *ἐν κυλινδρικῇ ἐπιφανείᾳ* Hu *ἐν κοχλοειδεῖ ἐπιφανείᾳ* Tr *ἐν πλεκτοειδεῖ ἐπιφανείᾳ* coniecit Ver Eecke
[3] *διά δε τῆς* A corr. To, Hu, Tr
[4] *πρὸς γραμμὴν* ABS *πρὸς γραμμῇ* Hu, Tr
[5] *τὴν ΔΘ περιφέρειαν* ABS corr. Co, Hu, Tr
[6] *η* (sine spirit. et acc.) A corr. Hu, Tr
[7] coniunx. Hu, Tr
[8] *περιφέρειαν* add. Hu
[9] *κινουμένη* A corr. Hu, Tr

παροδεύει τὴν ΑΔΓ[1] περιφέρειαν, ση-
μεῖον ἐπ' αὐτῆς ἀρξάμενον ἀπὸ τοῦ Β ἐπὶ τὸ *Γ*[2] *παραγενέσθω*[3]
θέσιν λαβούσης τὴν <ΓΒ τῆς>[4] ΑΒ, καὶ ποιείτω τὴν *ΒΗ Α*[5] ἕλικα. ἔστιν
ἄρα ὡς ἡ ΑΒ πρὸς ΒΗ, ἡ ΑΔΓ περιφέρεια πρὸς τὴν ΓΔ, καὶ
ἐναλλάξ. ἀλλὰ καὶ ἡ ΕΖ πρὸς ΔΓ· ἴση ἄρα ἡ ΒΗ τῇ ΖΕ. ἤχθω
τῷ ἐπιπέδῳ ὀρθὴ ἡ ΚΗ ἴση τῇ ΒΗ· ἐν κυλινδροειδεῖ ἄρα
ἐπιφανείᾳ τῇ ἀπὸ τῆς ἕλικος τὸ Κ. ἀλλὰ καὶ ἐν κωνικῇ,
ἐπιζευχθεῖσα γὰρ ἡ ΒΚ ἐν κωνικῇ[6] γίνεται ἐπιφανείᾳ
ἡμίσειαν ὀρθῆς κεκλιμένη[7] πρὸς τὸ ὑποκείμενον καὶ
ἠγμένη διὰ δοθέντος τοῦ Β· πρὸς γραμμῇ[8] ἄρα τὸ Κ. ἤχθω διὰ
τοῦ Κ τῇ ΕΒ παράλληλος ἡ ΛΚΙ, καὶ ὀρθαὶ τῷ ἐπιπέδῳ
αἱ ΒΛ ΕΙ· ἐν *πληκτοειδεῖ*[9] ἄρα ἐπιφανείᾳ ἡ ΛΚΙ, φέρεται γὰρ
διά τε τῆς ΒΛ εὐθείας θέσει οὔσης καὶ διὰ θέσει γραμμῆς πρὸς ᾗ
τὸ Κ· καὶ τὸ Ι ἄρα <ἐν> ἐπιφανείᾳ[10]. ἀλλὰ καὶ ἐν ἐπιπέδῳ, ἴση γὰρ
ἡ ΖΕ τῇ ΕΙ, ἐπεὶ καὶ τῇ ΒΗ, καὶ γίνεται παρὰ θέσει ἡ ΖΙ κάθετος
οὖσα ἐπὶ τὴν ΒΓ· πρὸς γραμμῇ[11] ἄρα τὸ Ι [σ][12], ὥστε καὶ τὸ Ε. καὶ δῆλον ὅτι
ἂν ὀρθὴ <ᾖ>[13] ἡ ὑπὸ ΑΒΓ γωνία, ἡ προειρημένη τετραγωνίζουσα γραμμὴ γίνεται.



[1] *τὴν ΑΔ περιφέρειαν* A corr. Co, Hu, Tr
[2] *ἀπὸ τοῦ Β ἐπὶ τὸ Γ* A *ἀπὸ τοῦ Α ἐπὶ τὸ Β* Hu, Tr
[3] *παργενέσθω* A *παραγενέσθω* To, Tr *παραγινέσθω* Hu
[4] ΓΒ *τῆς* add. Hu, Tr
[5] coniunx. Hu, Tr
[6] *γωνικῇ* A corr. S Hu, Tr
[7] *κεκλιμένης* AB corr. S Hu, Tr
[8] *πρὸς γραμμὴ* A Tr *πρὸς γεγραμμένην* B1 corr. B2 vel B3 To, Hu *πρὸς γραμμὴν* S
[9] *πλέκτοειδεῖ* Hu
[10] ἐπιφανεια (sine acc.) A *ἐπιφάνεια* BS *ἐν ἐπιφανείᾳ* To, Hu, Tr
[11] *προςγραμμη* A *πρὸς γραμμὴ* B *προσγραμμὴ* S corr. To, Hu, Tr
[12] *σ* del. Hu, Tr
[13] *ᾖ* add. Hu, Tr

Prop. 30
#35 ὥσπερ <ἐν>[1] ἐπιπέδῳ νοεῖται γι-
νομένη τις ἕλιξ φερομέ-
νου σημείου κατ' εὐθείας κύ-
κλον περιγραφούσης, καὶ
ἐπὶ στερεῶν φερομένου[2]

f. 49v (Prop. 30)
σημείου κατὰ μιᾶς πλευρᾶς *τὴν*[3] ἐπιφάνειαν περιγραφού-
σης, οὕτω *δὲ*[4] καὶ ἐπὶ σφαίρας ἕλικα νοεῖν ἀκόλουθόν ἐστι γραφο-
μένην τὸν τρόπον τοῦτον. ἔστω ἐν σφαίρᾳ μέγιστος κύκλος ὁ
ΚΛΜ περὶ πόλον τὸ[5] Θ σημεῖον, καὶ ἀπὸ τοῦ Θ μεγίστου κύ-
κλου τεταρτημόριον γεγράφθω τὸ ΘΝΚ, καὶ ἡ μὲν ΘΝΚ πε-
ριφέρεια, περὶ τὸ Θ μένον φερομένη κατὰ τῆς ἐπιφανείας
ὡς ἐπὶ τὰ Λ Μ[6] μέρη, ἀποκαθιστάσθω πάλιν ἐπὶ τὸ αὐτό,
σημεῖον δέ τι φερόμενον ἐπ' αὐτῆς[7] ἀπὸ τοῦ Θ ἐπὶ τὸ Κ πα-
ραγινέσθω· γράφει δή τινα ἐπὶ τῆς ἐπιφανείας ἕλικα,
οἵα ἐστὶν ἡ ΘΟΙΚ, καὶ ἥτις ἂν ἀπὸ τοῦ Θ γραφῇ μεγίστ<*ου κύκλου περιφέρεια, ἔστιν ὡς*> ἡ τοῦ[8]
κύκλου περιφέρεια, πρὸς τὴν ΚΛ <περιφέρειαν λόγον ἔχει ὃν>[9] ἡ ΛΘ
πρὸς τὴν ΘΟ· λέγω δὴ
ὅτι, ἂν ἐκτεθῇ[10] τεταρτημόριον τοῦ μεγίστου ἐν τῇ σφαί-
ρᾳ κύκλου *τ' οὗ* ΑΒΓ *περιφέρεια*, κέντρον τὸ Δ[11], καὶ ἐπι-
ζευχθῇ ἡ ΓΑ, γίνεται ὡς ἡ τοῦ ἡμισφαιρίου ἐπιφάνεια πρὸς
τὴν μεταξὺ τῆς ΘΟΙΚ ἕλικος καὶ τῆς ΚΝΘ περιφε-
ρείας ἀπολαμβανομένην[12] ἐπιφάνειαν, οὕτως ὁ ΑΒΓΔ
τομεὺς πρὸς τὸ ΑΒΓ τμῆμα. ἤχθω γὰρ ἐφαπτομένη τῆς
περιφερείας ἡ ΓΖ, καὶ περὶ κέντρον τὸ Γ διὰ τοῦ Α γε-
γράφθω[13] περιφέρεια ἡ ΑΕΖ· ἴσος ἄρα ὁ ΑΒΓΔ τομεὺς

[1] *ἐν* add. Hu, Tr
[2] *ε/////////// φερομένου* A *ἔπειτα φερομένου* voluit Co *ἐπὶ στερεῶν φερομένου* Hu *ἐν ὀρθῷ κώνου φερομένου* Tr
[3] *τὴν* A Tr *τιν'* Hu
[4] *οὕτω δὲ* A *οὕτως δὴ* Hu, Tr
[5] *περὶ πόλον τὸν Θ σημεῖον* A corr. Hu, Tr
[6] *τὰ ΛΑΜ μέρη* A corr. Co, Hu, Tr
[7] *ἀπ' αὐτῆς* A corr. Hu, Tr
[8] *μεγίστη τοῦ* A *–ου...ὡς* add. Tr *μεγίστου* Hu
[9] *λόγον ἔχει ὃν* add. Hu
[10] *ἐκτεθη* A corr. Hu, Tr
[11] *τοῦ* ΑΒΓ *περιφέρεια κέντρον* A *τὸ* ΑΒΓ *περὶ κέντρον* Hu *τ' οὗ ΑΒΓ* Tr
[12] *ἀπολαμβανομένης* ABS corr. Hu, Tr
[13] *διὰ τοῦ ΑΓ εγράφθω* A corr. Hu, Tr

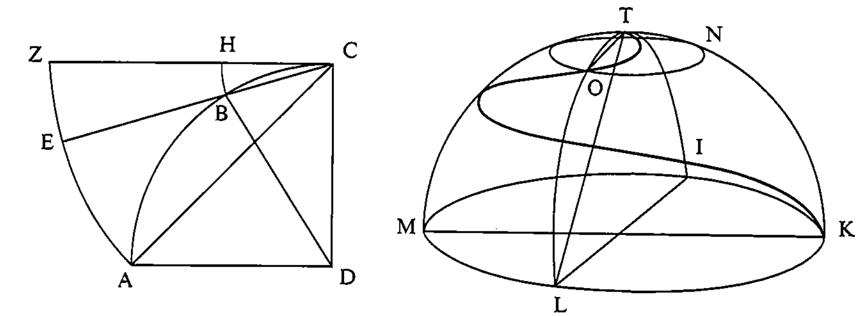
τῷ ΑΕΖΓ, διπλασία μὲν γὰρ ἡ πρὸς τῷ Δ γωνία τῆς ὑπὸ ΑΓΖ, ἡμίσυ δὲ τὸ ἀπὸ ΔΑ τοῦ ἀπὸ ΑΓ. ὅτι ἄρα καὶ ὡς αἱ εἰρημέναι ἐπιφάνειαι πρὸς ἀλλήλας, οὕτως ὁ ΑΕΖΓ[1] τομεὺς πρὸς τὸ ΑΒΓ τμῆμα. ἔστω μέρος[2] ἡ ΚΛ περιφέρεια τῆς ὅλης τοῦ κύκλου περιφερείας, καὶ τὸ αὐτὸ μέρος [*ὅδε μέρος*][3] ἡ ΖΕ τῆς ΖΑ, καὶ ἐπεζεύχθω ἡ ΕΓ· ἔσται δὴ καὶ ἡ ΒΓ τῆς ΑΒΓ τὸ αὐτὸ μέρος. ὃ δὲ μέρος ἡ ΚΛ τῆς ὅλης περιφερείας, τὸ αὐτὸ καὶ ἡ ΘΟ τῆς ΘΟΛ. καὶ ἔστιν ἴση ἡ ΘΟΛ[4] τῇ ΑΒΓ· ἴση ἄρα καὶ ἡ ΘΟ τῇ ΒΓ. γεγράφθω περὶ πόλον τὸν Θ διὰ τοῦ Ο περιφέρεια ἡ ΟΝ, καὶ διὰ τοῦ Β περὶ τὸ Γ κέντρον ἡ ΒΗ. ἐπεὶ οὖν ὡς ἡ ΛΚΘ σφαιρικὴ ἐπιφάνεια πρὸς τὴν *Ο Θ* Ν[5], ἡ ὅλη τοῦ ἡμισφαιρίου ἐπιφάνεια[6] πρὸς τὴν τοῦ τμήματος[7] ἐπιφάνειαν οὗ ἡ ἐκ[8] τοῦ πόλου ἐστὶν ἡ ΘΟ, ὡς δ' ἡ[9] τοῦ ἡμισφαιρίου ἐπιφάνεια πρὸς τὴν τοῦ τμήματος ἐπιφάνειαν, οὕτως ἐστὶν τὸ ἀπὸ

f. 50 (Prop. 30 and metatheoretical passage)

τῆς τὰ *ΘΛ*[10] ἐπιζευγνυούσης εὐθείας τετράγωνον πρὸς τὸ ἀπὸ τῆς ἐπὶ τὰ *ΘΟ*[11], ἣ τὸ ἀπὸ τῆς ΕΓ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΒΓ, ἔσται ἄρα καὶ ὡς ὁ ΚΛΘ τομεὺς ἐν τῇ ἐπιφανείᾳ <πρὸς>[12] τὸν ΟΘΝ, οὕτως ὁ ΕΖΓ τομεὺς πρὸς τὸν ΒΗΓ. ὁμοίως δείξομεν ὅτι καὶ <ὡς>[13] πάντες οἱ ἐν τῷ ἡμισφαιρίῳ τομεῖς οἱ ἴσοι τῷ ΚΛΘ, οἵ εἰσιν ἡ ὅλη τοῦ[14] ἡμισφαιρίου ἐπιφάνεια πρὸς τοὺς περιγραφομένους περὶ τὴν ἕλικα τομέας ὁμοταγεῖς τῷ ΟΘΝ, οὕτως <πάντες>[15] οἱ ἐν τῷ ΑΖΓ τομεῖς οἱ ἴσοι τῷ ΕΖΓ, τουτέστιν ὅλος ὁ ΑΖΓ τομεύς, πρὸς τοὺς περιγραφομένους περὶ τὸ ΑΒΓ τμῆμα τοὺς[16] ὁμοταγεῖς τῷ

[1] *ΑΕΓΖ* AS corr. B Hu, Tr
[2] *ὅ μέρος* Hu
[3] *ὁ δὲ μέρο ἡ* A *ὅδὲ μέρος ἡ* B *ὅδὲ μερη* S *ὁ δὲ μέρος* del. Tr *περιφέρεια* Hu
[4] *ΘΟΑ* AB2S corr. B1 Hu, Tr
[5] coniunx. Hu, Tr
[6] *πρὸς...ἐπιφάνεια* add. A2 in margine
[7] *τὴν τοῦ ἡμισφαιρίου* ABS *τὴν τοῦ τμήματος* Co, Hu, Tr *τὴν ἐντὸς τοῦ ἡμισφαιρίου* coni. Hu
[8] *οὐκ ἐκ* A corr. Co, Hu, Tr
[9] *δὴ* A *δ' ἡ* Hu, Tr
[10] distinx. B1S Hu, Tr
[11] distinx. B Hu, Tr *Θ Ο, τουτέστιν* Co
[12] *πρὸς* add. Hu, Tr
[13] *ὡς* add. Hu, Tr
[14] *οιείσιν οι ολη* A corr. S Hu
[15] *πάντες* add. Hu, Tr
[16] *τοὺς* A Hu *τομέας* Tr

ΓΒΗ. τῷ δ' αὐτῷ τρόπῳ δειχθήσεται καὶ ὡς ἡ τοῦ ἡμισφαιρίου <ἐπιφάνεια>[1] πρὸς τοὺς ἐγγραφομένους τῇ ἕλικι τομέας, οὕτως ὁ ΑΖΓ τομεὺς πρὸς τοὺς ἐγγραφομένους τῷ ΑΒΓ τμήματι τομέας, ὥστε καὶ ὡς ἡ τοῦ ἡμισφαιρίου ἐπιφάνεια πρὸς τὴν ὑπὸ τῆς ἕλικος ἀπολαμβανομένην ἐπιφάνειαν, οὕτως ὁ ΑΖΓ τομεύς, τουτέστιν [ὡς][2] τὸ ΑΒΓΔ τεταρτημόριον, πρὸς τὸ ΑΒΓ τμῆμα. συνάγεται δὲ διὰ τούτου ἡ μὲν ἀπὸ τῆς ἕλικος ἀπολαμβανομένη ἐπιφάνεια πρὸς τὴν ΘΝΚ περιφέρειαν ὀκταπλασία τοῦ ΑΒΓ τμήματος, ἐπεὶ καὶ ἡ τοῦ ἡμισφαιρίου ἐπιφάνεια τοῦ ΑΒΓΔ τομέως, ἡ δὲ μεταξὺ τῆς ἕλικος καὶ τῆς βάσεως τοῦ ἡμισφαιρίου ἐπιφάνεια ὀκταπλασία τοῦ ΑΓΔ τριγώνου, τουτέστιν ἴση τῷ ἀπὸ τῆς διαμέτρου τῆς σφαίρας τετραγώνῳ.



Metatheoretical passage

#36 τὴν δοθεῖσαν γωνίαν εὐθύγραμμον εἰς τρία ἴσα τεμεῖν οἱ παλαιοὶ γεωμέτραι θελήσαντες ἠπόρησαν δι' αἰτίαν τοιαύτην. τρία γένη φαμὲν εἶναι τῶν ἐν γεωμετρίᾳ προβλημάτων, καὶ τὰ μὲν αὐτῶν ἐπίπεδα καλεῖσθαι, τὰ δὲ στερεά, τὰ δὲ γραμμικά.

f. 50v (metatheoretical passage)

τὰ μὲν οὖν δι' εὐθείας καὶ κύκλου περιφερείας δυνάμενα λύεσθαι *λέγοιτο* ἂν[3] εἰκότως ἐπίπεδα· καὶ γὰρ αἱ γραμμαὶ δι' ὧν εὑρίσκεται τὰ τοιαῦτα προβλήματα τὴν γένεσιν ἔχουσιν ἐν ἐπιπέδῳ. ὅσα δὲ λύεται προβλήματα παραλαμβανομένης εἰς τὴν *γένεσιν*[4] μιᾶς τῶν τοῦ κώνου τομῶν ἢ καὶ πλειόνων, στερεὰ ταῦτα κέκλη-

[1] *ἐπιφάνεια* add. Hu, Tr
[2] *ὡς* ABS del. Hu, Tr
[3] *λέγοιτ' ἂν* Hu
[4] *εἰς τὴν γένεσιν* ABS *εἰς τὴν κατασκευὴν* Co *εἰς τὴν εὕρεσιν* Hu

excerpt and present that part of it that concerns two mean proportionals for cube duplication. Then the apparent contradiction is diminished, and Nicomedes is acknowledged as the source for Prop. 24. The degree to which Pappus edited his source cannot be determined with certainty, unless one can find evidence for Eutocius' independence from Pappus. I will treat Prop. 24 as essentially Nicomedean.

The proof protocol for Prop. 24 will be given in some detail, to illustrate the fact that Nicomedes' methods, within *symptoma*-mathematics, were quite "standard," unlike, e.g., Archimedes' procedure in Prop. 21, and correspond in scope to the means employed in Apollonius' analytical works.

5.3.3.1 Proof Protocol Prop. 24

1. Protasis
   Find two mean proportionals for CL, LA
2. Ekthesis/Kataskeue
   Rectangle ABCL, construct, M, Z, K, TK
   CK, MA solve the problem, i.e.: CL:CK = CK:MA = MA:AL
3. Apodeixis:
   Show that BM:BK can be expressed in three ways

3.1 MB:BK = CK:MA	
$BK \times KC + CE^2 = EK^2$	[II, 6]
$BK \times KC + CZ^2 = KZ^2$	[I, 47]
MA:AB = ML:LK	[VI, 2 with V, 16]
ML:LK = BC:CK	[VI, 2]
MA:AB = BC:CK;	
MA:AD = HC:CK; HC:KC = ZT:TK	[VI, 2]
MA:AD = ZT TK;	
MD:DA = ZK:TK	[V, 18]
DA = TK $\Rightarrow$ MD = ZK	[V, 9]
$MD^2 = BM \times MA + DA^2$	[II, 6]
$BM \times MA + DA^2 = ZK^2 = BK \times KC + CZ^2$;	
$DA^2 = CZ^2$	
$\Rightarrow BM \times MA = BK \times KC$	
$\Rightarrow$ BM:BK = KC:MA	[VI, 16].
3.2 MB:BK = LC:CK	[VI, 4]
3.3 MB:BK = MA:AL	[VI, 4]

4. The equations 3.1–3.3 establish
   CL:CK = CK:MA = MA:AL.

5.3.4 Prop. 25: Cube Multiplication in a Given Ratio

Set the ratio out as a:b. Via Prop. 24, construct c, d so that a:c = c:d = d:b. Then $a:b = (a:c)^3 = a^3:c^3$.

Note that triple ratios are here identified with ratios of cubes. V, def. 11: $(a:c)^3 = a:b$. According to XI, 33, the cubes stand to each other in the same triplicate ratio.

5.4 *Props. 26–29: Quadratrix/Squaring the Circle*

5.4.1 General Observations on Props. 26–29

5.4.1.1 Structure of Props. 26–29

Genesis and *symptoma* of the quadratrix as a motion curve
Sporus' criticism of the quadratrix (specifically of the *genesis*)
Props. 26 and 27: *symptoma*-mathematics of the quadratrix: rectify and square the circle.
Props. 28 and 29: geometricize the *genesis* of the quadratrix via analysis on surfaces.

context: motion curves and *symptoma*-mathematics, squaring the circle.
sources: Nicomedes or Dinostratus on quadratrix, Sporus' *Aristotelian Wax Tablets* for criticism of the *genesis*, Nicomedes (?) for exhaustion proof in Prop. 26[1]; unknown sources for Props. 28 and 29 (Apollonius? Nicomedes?).
means: I, II,V,VI, *Circ. mens.* 1 for Props. 26 and 27;
no recourse to the *Elements* in Props. 28 and 29.
method: exhaustion proof, synthetic (Prop. 26), analysis (Props. 28 and 29).
format: non-uniform: *genesis* and *symptoma* is descriptive; Sporus' criticism is an excerpt from a philosophical refutation argument, rhetorically styled; Prop. 26 is a theorem, Prop. 27 a problem; Props. 28 and 29 give an analytical determination of a curve.
reception/historical significance: the quadratrix was much discussed in the seventeenth century, as an example for a non-geometrical, or a transcendent, curve.
embedding in *Coll.* IV: connection to the plane spiral (Prop. 19): Props. 25, 26 and 29, motif "author Nicomedes": Props. 23–25; motif "*genesis* via synchronized motions": Props. 19, 30; motif "linear problems and *symptoma*-mathematics of the quadratrix": Props. 35–41; motif "analytical interpretation of the *symptoma*": Prop. 23.
purpose: exemplary illustration of the problems, and the mathematical potential of curves of the third (linear) kind.
literature: Heath (1921, I, pp. 225–230), Knorr (1986, pp. 80–88; 226–233; 166–167) for Props. 26 and 27; on Props. 28 and 29 and its context of analysis on surface loci see Heath (1921, I, 439–440; II, pp. 380–382), Knorr (1978a, pp. 62–66, 1986, pp. 129 and 166–167), Jones (1986a, pp. 595–598), Ver Eecke (1933b, pp. 197–201), Chasles (1875, pp. 30–37) and Notes VIII. *Coll.* VII, pp. 1004–1014 Hu (Jones 1986a, pp. 362–371) contain Pappus' commentary on Euclid's work in loci on surfaces (probably conics). The work he comments on probably rested on prior contributions by Aristaeus.

5.4.1.2 Authorship for Props. 26–29

Pappus mentions both Dinostratus and Nicomedes as authors that used the quadratrix in connection with the squaring of the circle. For biographical information on

[1] Pappus himself can be excluded as the author of Prop. 26, because he explicitly says he is reporting.

Nicomedes see above, introduction to Props. 23–25. Dinostratus was the brother of Menaechmus (inventor of conic sections). He lived ca. 350 BC and may very well have been a pupil of Eudoxus. We know almost nothing about his mathematical work outside of our passage in *Coll.* IV, so what will be said here is to some degree speculative.

Pappus clearly associates both him and Nicomedes with the use of the quadratrix for squaring the circle. Since the fifth century BC sophist Hippias is mentioned elsewhere as the inventor of the curve itself, perhaps Hippias used the curve for the angle trisection (indeed: arbitrary division), while Dinostratus discovered its rectification property, and possibly proved it with the Eudoxean method of exhaustion (not Prop. 26, however). After Archimedes, due to *Circ. mens.* I, the quadratrix could then have been employed by Nicomedes to square the circle (Prop. 27). Perhaps it was Nicomedes, also, who is responsible for the proof of the rectification property in its present form. The proof of Prop. 26 as given in Pappus is almost certainly post-Archimedean, because it relies implicitly on a theorem that is equivalent to Arch., *Circ. mens.* 1 (see proof protocol below). Furthermore, Nicomedes could have pursued the analytical *symptoma*-approach to the properties of the curve, and be at least partially responsible for the analytical reduction of the quadratrix, viewed as *symptoma*-curve, to Archimedes' spiral (Prop. 29). In analogy to the conchoid, which is determined pointwise as a kind of *neusis*-curve, the quadratrix can be seen as a curve corresponding, at each point, to the correlation of the very same rotation + linear motion used in the plane spiral (when it is inscribed in a circle), and such a perspective leads to the way the quadratrix is characterized in Prop. 29. On this view, the contribution of Dinostratus is substantial, but Nicomedes would be the one who developed the *symptoma*-mathematics of the curve theoretically, as a member of the class of higher curves.

This is, as it were, the maximum option for Dinostratus' and Nicomedes' achievements in relation to the quadratrix. I am putting it forth tentatively. It has the advantage of fitting well with a sympathetic reading of Pappus' text, and it can account for the fact that in most later sources, it is only Nicomedes that is associated with the study of the properties of the quadratrix as a higher curve.[1] In what follows, I will briefly sketch two alternative views.

Heath (1921) would like to ascribe much of the mathematics on the quadratrix to Hippias already, including the discovery of the rectification property and its proof via exhaustion. Perhaps this is a little too optimistic for Hippias.[2] It would place a considerable amount of mathematical theory and expertise already in the fifth century BC, and is therefore somewhat unconvincing. Knorr (1986) has argued a different view. He denies that Hippias could have had anything to do with the

[1] Cf. Iamblichus apud Simpl. in Cat. 192 Kalbfleisch, 645 b Brandis. Procl. in Eucl. 272 Friedlein mentions only Nicomedes as well, but not Dinostratus, although Proclus must have known about him. His name had been mentioned earlier, in the catalog of mathematicians derived from Eudemus.

[2] Hippias would be credited with an expertise in handling the exhaustion method that is in this form usually associated with Eudoxus, who lived considerably later than Hippias, and this reading also would leave no room, as it were, for Dinostratus.

curve, because it involves a considerable degree of mathematical sophistication and expertise. Rather, according to Knorr, Dinostratus may have invented the curve and discovered its rather obvious angle division property (i.e., he ascribes to Dinostratus what others had ascribed to Hippias). Because the *genesis* of the curve via motions has strong affinities with the Archimedean plane spiral (inscribed version). Knorr believes that in the generation after Archimedes this connection was made use of, by Nicomedes probably, to formulate, and to prove the rectification property of the curve (Prop. 26 entirely), whence the quadrature of the circle follows as a corollary thanks to *Circ. mens.* I. The rectification property is implicit in the curve, but unlike the angle section property, it cannot be read off directly. It seems plausible to assume that the latter was discovered after the former. As for Props. 28 and 29, Knorr envisages Apollonius as a possible source for Prop. 28 (because of the central role of the Apollonian helix therein), and associates Prop. 29 with Archimedes.

Perhaps this is a bit too pessimistic with regard to Hippias and Nicomedes. There seems to be no compelling reason to discard the unequivocal testimony that Hippias invented the curve itself. How much he knew about it may be uncertain, but the angle section property is indeed easily deduced. Archimedes seems unlikely as a source for a predominantly analytical investigation of motion curves as in Prop. 29. In his other works, and in his heuristic method, Archimedes shows no preference for the analytic approach. His interests point rather in the direction of quasi-mechanical methods, and perhaps infinitesimals. Prop. 29 appears not to be Archimedean in style. More plausible is the connection between Prop. 28 and Apollonius, because Apollonius did in fact favor, and develop, the analytical approach in geometry, and is said in Iamblichus to have called the quadratrix "sister of the cochlias." This somewhat enigmatic statement could be read as a description of Prop. 28, and then Apollonius could be its author.[1] Finally, Apollonius may have written a treatise on the helix. The evidence on such a treatise is slim, however. I prefer to refrain from ascribing the substance of Prop. 28 to him directly, while supporting the claim that Prop. 28 is well in line with higher mathematics, Apollonian style, i.e., in a tradition developing an approach to mathematics that is exemplified in Apollonius. As said above, I am inclined to assign to Nicomedes the leading role in the shaping of Props. 26 and 27, and to consider a substantial contribution to Prop. 29 on his part as a distinct possibility. Furthermore, I am of the opinion that the upcoming minor results on the *symptoma*-mathematics of the quadratrix in Props. 35–41 may derive from his treatise on the quadratrix as well.

5.4.1.3 Quadratrix

As outlined above, the quadratrix is usually associated with Hippias of Elis (fifth century BC) as its inventor. It was at first used to trisect the angle; in fact it can

[1] Iambl. apud Simpl. in Cat. 192, 19–24 Kalbfleisch. Heath (1921, I. p. 225) supports a different view.

divide an angle in any ratio. Later on, it was discovered that it can also be used to rectify the circle, and thus to solve the problem of squaring the circle. This solution via the quadratrix was not accepted as a constructive solution, because in setting out the quadratrix one already has to assume access to the ratio of diameter and circumference of the circle (see below). Nevertheless, Pappus was willing to fully accept the mathematics on, or about, the quadratrix as an example of geometry of what he called the "linear" kind. This applies to Props. 26–29, and to Props. 35–41. The status of the curve itself was left somewhat in limbo by him, and his ambivalent portrait probably contributed to the fact that, in the seventeenth century, the quadratrix was used as one of the primary examples of curves that did not fit the bill of Descartes' definition of a proper geometrical curve,[1] and might therefore be used as either a vantage point to enlarge, or a counter-example in the attempt to delimit the horizon of geometry and analysis.[2]

Props. 26 and 27, with their prefatory detailed description of the quadratrix as a motion curve are our only surviving evidence on this curve (and the squaring of the circle with it) from antiquity. Props. 35–41 show us the use of the quadratrix for the general angle division, and results derived from it, as well as further properties following from the rectification property. Again, those are the only such sources extant from antiquity. Finally, Props. 28 and 29 are our only extant detailed examples for an analysis of loci on surfaces (used here for the geometrical justification/description of the *genesis* of the quadratrix). Obviously, this makes Props. 26–29 (to a lesser extent: Props. 35–41) a document of the highest importance for the history of ancient mathematics. However, the fact that no other ancient source gives such a detailed insight into the discussion of this curve and its geometrical properties, and that there are no traces of parallel accounts on other higher curves, also entails, unfortunately, that we have no context in which to set, and from which to evaluate, Pappus' portrait of the curve and its mathematics. There seems to have been a mathematical community (or just a small group of mathematicians?) who pursued this kind of mathematics for some time (how long? just one generation, or 100 years?). Pappus and others list names. What did these mathematicians think they were doing? What was their view on the status of curves like the quadratrix, and on the *symptoma*-mathematics on them? What was the mainstream view (if such a view existed) on this collection of mathematical treatises? Is Pappus' rational reconstruction of the quadratrix and the degree to which it can be "geometricized" representative, and if so: representative of what? Wherever his pronouncements

[1] In Descartes ed. Schooten (1659, pp. 18 and 38), for example, the quadratrix and the spiral appear as the primary examples for non-geometrical curves.

[2] I am not aware of any systematic study of the evidence. Such a project would seem to me to be rather promising, because the amount of available source texts is rather extensive and widespread. The quadratrix does turn up, e.g., in Jacob Bernoulli's papers (he was also interested in spiral lines), and also in Leibniz's mathematical manuscripts. Leibniz borrowed the name and applied it to a more general type of curve with "quadrature" properties. For the Cartesians, the quadratrix was a classic example for a non-permissible curve.

remain vague, should we conclude that he is uncertain or does not understand, or that the discussion had not reached a consensus, or that there was no discussion and the project just died out? Perhaps a detailed comprehensive comparative study of Props. 19–30 and 35–41 in connection with *SL*, taking into account also the scattered summary remarks in other authors (especially Proclus and the commentators on Aristotle) could shed some new light on this issue. It cannot be pursued here. What is given is a documentation of the ancient evidence on the mathematics of the quadratrix and on Pappus' evaluation of it, as far as the full text in *Coll.* IV attests it. Perhaps this material could be the basis for further investigations on a broader scope.

5.4.1.4 Squaring the Circle

The squaring of the circle, i.e., the problem of finding a geometrical construction to transform a given circle into a square, caught the attention of the Greeks very early on. Already in the fifth century BC, they appear to have found a (very elementary) way of transforming any given polygon into a square – II, 14 in the *Elements*, resting on I, 44 and I, 45. It seemed that an analogous procedure to do the same with the circle should be possible. The question captured the imagination of mathematicians and non-mathematicians alike, and it sparked the development of new methods and new theories in geometry, with the goal to become able, among other things, to solve this problem with the new mathematics. Already in Aristotle's time doubts arose as to whether the problem was solvable at all. But the discussion, and the search, continued nevertheless, and it continued beyond antiquity. In a sense, the matter was finally settled only with Lindemann's proof of the transcendence of π in 1882. No construction with means that are equivalent to the solution of an algebraic equation with rational coefficients is possible. One needs infinitesimal methods, or else a curve like the quadratrix. In what follows, a survey of the attested ancient attempts at solving the problem, and a selective list of later attempts and judgments is given.[1]

Hippocrates of Chios, in the fifth century BC succeeded in constructing three out of the five quadratures of lunulae that are possible within plane geometry. One of them is located over a semicircle, one over a segment that is larger than the semicircle, and one over a segment that is smaller. He also squared a figure composed

[1] For further information on the ancient quadratures cf. Heath (1921, I, pp. 183–201, 220–235) and Knorr (1986) passim. Knorr is perhaps not always careful in demarcating textual/historical evidence from his own reconstructions. Tropfke (1923, pp. 195–238) provides a survey of attempts to square the circle, focusing on contributions known in Western Europe. It is still valuable for its numerous bibliographical references. Also worth reading, though in some respects outdated, is Rudio (1892), a monograph on the measurement of the circle that prints the major contributions by Archimedes, Huygens, Lambert, and Legendre in full, and also contains a survey on the history of the quadrature.

of a circle and a lunula. His constructions probably were intended as steps toward squaring the circle. As Aristotle points out, they were fully valid mathematical arguments, but they do not square the circle.[1] Still in the fifth century, the sophists Antiphon and Bryson presented arguments that they held to be solutions to the problem of squaring the circle.[2] Both started with an inscribed square and constructed a sequence of polygons approximating the circle more closely at each step. Bryson used, in addition, a corresponding sequence of circumscribed polygons. Both assumed that in the process, the circle is exhausted.[3] Antiphon in effect assumed that the process would be finite, and that the circle coincides with a polygon having very small sides. This polygon can then be squared, via the equivalent of II, 14. His argument was considered as invalid mathematically, because it rests on a non-geometrical concept of a circle.[4] Bryson did not identify the circle with a polygon. He argued that since all the "inside" polygons are smaller than the circle, and all the "outside" ones are larger, there must exist a polygon that has the same area as the circle. Such a polygon could, again, be squared via the equivalent of II, 14. Bryson claimed thus to have squared the circle. His argument rests on the idea that "area," taken abstractly, is a continuous quantity. Though not invalid in itself, it was taken by Aristotle not to be a geometrical argument at all, because it violates his homogeneity criterion.[5] Whether or not the mathematicians would have shared Aristotle's opinion here, it is clear that the argument does not amount to a geometrical solution to the (construction!) problem of squaring the circle. The required square – though Bryson's argument may reassure us that it exists – cannot actually be produced from this argument. Bryson did not square the circle. The problem thus was still unsolved in Aristotle's times, and he uses it frequently as an illustration for failed or as yet unsuccessful attempts in scientific inquiry. In his writings, one

[1] Heath (1921, I, pp. 183–201); cf. Simpl. in Phys. 56–68 Diels, Rudio (1907). See also Aristotle on Hippocrates' quadratures in Heath (1970, originally: 1949). The above judgment is taken from Aristotle, but there is no reason to assume that the mathematicians would not have shared his opinion.

[2] Cf. Heath (1921, I, pp. 221–225).

[3] Note that this does not mean that they used the Eudoxean method, the so-called "method of exhaustion." That method is, in its essence, a double reductio argument. Even though it was often used in connection with area and volume theorems, and in a context where a process of approximation is assumed, such a process is not essential to the method as such. That is: although it was used for arguments that we translate into limit arguments – and for others, too – it was in itself not a concealed limit argument. Heath is mistaken in assuming that Antiphon's and Bryson's "quadratures" contain the nucleus of the famous Eudoxean method, even though they may have anticipated infinitesimal procedures to some degree.

[4] This argument, found in Aristotle, again, would in all likelihood have been shared by the mathematicians.

[5] *Anal. Post.* I, 9. Aristotle differentiates between Antiphon's and Bryson's attempts. It is perhaps interesting that he does not reject Bryson's argument in itself as invalid (as he does in Antiphon's case), but rejects it as involving a "katabasis eis allo genos," as being ungeometrical, because of its failure to differentiate between geometrical and other continuous quantities. For Aristotle, arguments in geometry have to address geometrical entities qua geometrical, and not qua something else that they may also be.

also finds the suggestion that the squaring of the circle may be impossible in principle, because straight line and circle are generically different – even though it's clear, on the basis of continuity assumptions, that a straight line with a square equal to the circle must exist.[1]

Alongside the squaring, rectification became an issue soon. Is the ratio of circumference to diameter expressible as a ratio of numbers? With Archimedes's work (*Circ. mens.* I), it became apparent that and how the rectification and the quadrature entail each other.[2] Archimedes' investigation of the plane spiral in the heuristic version could, taken together with a theorem like *SL* 18[3] and *Circ. mens.* I, be used for the squaring of the circle. Archimedes never presented such an argument, and in effect replaced the *genesis* of the spiral with one that avoids the problems the quadratrix and the original (inscribed) spiral have. With the spiral in this description, one can indeed no longer square the circle. Archimedes also provided, by means of logistics, two different approximations for π[4] (*Circ. mens.* II, III). Apollonius, Sporus, and Ptolemy gave further approximations for π, closer in numerical value than Archimedes'.[5] Dinostratus/Nicomedes used the quadratrix for circle rectification, and Nicomedes applied *Circ. mens.* I to obtain a "quadrature" (Props. 26 and 27). As noted above, this is not a constructive quadrature in the sense required, either, because the setting out of the quadratrix involves the ratio of circumference and diameter (essentially π), and that was the equivalent to what was sought in rectification. The squaring of the circle with the quadratrix, insofar as it is geometrical, is purely *symptomatic*.

5.4.1.5 Circle Quadrature Through the Ages

The following selective list of examples is intended to give an impression of the different results, perspectives, and methods developed within the horizon of

[1] Descartes would later repeat that same general statement in Descartes (1637, pp. 340/341) (90/91 Smith/Latham). He took it for granted that the circle cannot be squared, because circle and straight line belong to different, incomparable kinds.

[2] *Circ. mens.* I implies that the problem of squaring the circle can be reduced to the problem of rectifying the circumference; cf. Knorr (1986, p. 159).

[3] *SL* 18 is a *symptoma*-theorem on the spiral with circumscribed circle. It shows that in such a situation, the circumference is equal to the subtangent of the spiral at the endpoint of the first rotation. It does not yield a constructive rectification of the circle. It also does not provide a constructive solution for finding the tangent to a spiral of first rotation (cf. Vieta, *Varia responsa*, including an approximate construction for such a tangent).

[4] The name π was not used by the ancients. It was coined in the early seventeenth century by Ludolph van Ceulen (*Fundamenta geometrica* ed. W. Snel, Würzburg 1615). According to Tropfke (1923, p. 232), its first occurrence is even later: 1706, in Jones' *Synopsis palmariorum matheseos*. The label refers to a number, in modern terms. The ancients had a very different view on numbers and ratios. I am using π merely as an abbreviation here.

[5] Cf. Heath (1921, II, pp. 232–235) and Knorr (1986, pp. 155–159) for Archimedes' and other approximations; see also Heath (1921, I, pp. 180–189).

Western European culture over the centuries in connection with the squaring of the circle. It is based on the above-mentioned Tropfke (1923, pp. 198–238). The Indian mathematician Aryabhata, fifth century AD gave an astonishingly close approximate value for π.[1] His methods are interesting because of their combination of geometry, algebra, and what would in Greek terminology be called logistics. There must have been substantial contributions in Islamic culture, but they are not accessible to me. They did have an influence on Fibonacci's contribution.[2] During the Middle Ages, Archimedes' approximations were widely used, and they came to be regarded as exact by many (especially the simpler one: 22/7). Exact geometry, as a demonstrative science, did not receive much attention during the Middle Ages in Europe. An interesting use of mathematical motifs for philosophical purposes (in connection with infinity) including a treatment of the circle and π can be found in Cusanus.[3] In the Renaissance, Leonardo squared the circle by rolling up a cylinder with a base equal to the circle that is to be squared, and appealing to Arch., *Circ. mens.* I.[4] Stifel devised a mechanism using levers and scales to "weigh" π.[5] In the modern era, when algebraic and improved calculation methods became available, we find approximations of π by Vieta and Huygens, still operating within the framework of classical geometry and logistics. They developed Archimedes' basic approach via inscribed polygons, and refined the limits for the approximations.[6] Gregory, Newton, and Leibniz employed infinite series,[7] and Wallis used his arithmetic of infinites to characterize π. Leibniz called his result an "arithmetical quadrature of the circle."[8] In the eighteenth century (1766), Lambert used continuous fractions, and showed that π was an irrational number. Euler studied both π and e in connection with trigonometric functions.[9] Toward the end of the nineteenth

[1] Cf. Elfering (1975). The work has been translated into English.

[2] Cf. Tropfke (1923, pp. 211/212) for references; Fibonacci's value for π is 864/275, ca. 3.141818.

[3] Cf. Tropfke (1923, pp. 213/214). On Cusanus' studies in connection with the quadrature of the circle see also Hofmann (1990, I, pp. 47–77, II, pp. 179–192, 351–395).

[4] Cf. Tropfke (1923, p. 214), Cantor II^2, pp. 301–302.

[5] Cf. Tropfke (1923, p. 215); on Stifel see also Hofmann (1990, II, pp. 78–109).

[6] Cf. Tropfke (1923, pp. 215–216, 218–219); e.g., Vieta, in *Variorum de rebus mathematicis liber* VIII, p. 392 in the 1646 Schooten edition gives a value for π that is exact in the first nine digits, and uses the "Archimedean" approach via inscribed polygons. For Huygens's contribution, see the appendix to his *Theorenata de quadratura hyperboles, ellipseos et circuli ex Data portionum gravitatis centro*, Leiden 1651, and his *De circuli magnitudine inventa*, Leiden (1654), both in Ch. Huygens, *Varia Opera*, Leiden (1724, pp. 328–340, 351–387); see also Rudio (1892). The latter work also contains a full German translation of Archimedes's, Lambert's, and Legendre's treatment.

[7] Gregory's double series is equivalent to an approximation via arctanx, Leibniz's series for π/4 converges very slowly, so that it only has theoretical value. Newton used the series for arcsinx, Brouncker developed Wallis' solution into an infinite fraction.

[8] Cf. Tropfke (1923, pp. 223–230).

[9] Cf. Tropfke (1923, p. 229).

century, Lindemann proved the transcendence of π.[1] As said above, Lindemann's result means that a constructive quadrature is impossible with circle and straight line, with conics, with any curve expressible as a polynom with rational coefficients. One needs a curve like the quadratrix. In the end, Pappus' Prop. 26 and Prop. 27 are as good as it gets.

5.4.2 *Genesis* and *Symptoma* of the Quadratrix

Genesis: Start with a quadrant BAD in a square ABCD (clockwise) over the radius; use two motions: of BC along BA, parallel to AD, and of AB along the arc BD, synchronized so that they both reach the position of AD at the same time; during the process they create an intersection line BH, the quadratrix.[2]

Symptoma: As can be seen from the *genesis*, for any line AZE drawn to the curve and extended to the circumference, we get:

arc BD:arc ED = BA:ZT.

5.4.3 Criticism of the *Genesis* by Sporus

Sporus of Nicaea, ca. 200 AD was not a mathematician. Rather, he seems to have been a philosopher interested in epistemology and theory of science. Of his work *Aristotelian Wax Tablets* only fragments remain. They contain reflections on mathematical arguments from the standpoint of an Aristotelian theory of science.[3] His criticism of the *genesis* of the quadratrix is as follows.[4]

[1] Cf. (Tropfke 1923, pp. 231–232). Lindemann's proof for the transcendence of π (Lindemann 1882) is modeled on Hermite's proof for the transcendence of e. For ensuing improvements and simplifications of this proof cf. Tropfke loc. cit.

[2] Note the close connection to the *genesis* of the spiral as given in Prop. 19.

[3] On Sporus cf. Tannery (1912, I, pp. 178–184); the main source for our information on Sporus outside this passage in Pappus and the one mentioned below, in Eutocius, are the scholia on Aratus' *Phaenomena*. A further example for a criticism by Sporus, also in connection with the squaring of the circle, is found in *Eutoc. in Arch. Circ. mens.* III, 258–259 Heiberg. Sporus insists that Archimedes' approximate values are not exact, discusses the decisive difference between an exact and an approximate value, and produces a closer approximation than Archimedes' to show that it was not the exact value. Apparently around 200 AD already the nature of an approximation was not properly understood by some, who believed that a value – a ratio in numbers – must be true, i.e., correct.

[4] As mentioned in the survey table of Props. 26–29, the style of this passage is decidedly different from the rest of *Coll.* IV. We are clearly dealing with an argument from a philosophical work, in polemical style, one in which objections were raised against another position on epistemological grounds. Note, e.g., the rhetorical questions and the device of a ficticious dialogue; cf. translation.

(i) The definition via synchronized motions contains a petitio principii: to coordinate the rotation and the linear motion, you need the ratio of arc BD to AB – essentially π – the very thing the quadratrix was supposed to provide.[1]
(ii) Even objection (i) aside, the genetic definition does not capture the endpoint of the curve, because the intersection stops right when the moving lines coincide with AD. This endpoint, however, was needed for the rectification of the quadrant (Prop. 26, see below). Infinitely many other points on the curve are in fact constructible, e.g., all points that one would get by successive division of angle and radius in half. But the endpoint is not among them.
(iii) The endpoint of the curve cannot be interpolated by extending the line in the manner of producing a straight line, because the curve does not have a fixed direction (as the straight line does). In fact, the quadratrix does not even have a constant curvature.

Sporus concludes: as long as the ratio of circle and radius is unknown, or not given, the curve cannot be accepted. Pappus will in effect pick up right here in Props. 28 and 29, and show that it is *given* in the specific sense of geometrical analysis, if a helix, or the spiral, is granted. Whereas Hultsch (and others around 1900) dismissed Sporus' objections, most modern interpreters accept them as valid.[2] The curve is not well-defined. Note that the reason for Sporus' objections is not the use of motions as such, but the conceptual inconsistencies involved in this particular motion description. These inconsistencies will have to be circumvented, or abolished, if Sporus' objections to the mathematical use of the curve are to be met. And Pappus explicitly agrees with Sporus' reasons for rejecting the curve under the motion description (under the description as "mechanical"). He uses the word "eulogos" (with good reason). On the other hand, he insists that the argument about the quadratrix – Props. 26 and 27, the *symptoma*-quadrature – is "much more acceptable" mathematically. In Props. 28 and 29, Pappus will provide a geometrical analysis for the generation of the curve, via analysis of loci on surfaces. It is intended to meet, or rather perhaps to circumvent, the objections raised by Sporus, so as to "geometricize" the curve as a basis for valid *symptoma*-mathematics (see below).

5.4.4 Prop. 26: Rectification Property of the Quadratrix

5.4.4.1 Proof Protocol Prop. 26

This proof protocol is given in detail, because its content is a "classic," and also because it is the only example of a full-fledged argument via double reductio in *Coll.* IV (the other example in Prop. 13 is much less complex).

[1] As in the case of the spiral in the version given in Prop. 19, you need π to determine the speeds involved.

[2] Cf. Heath (1921, I, 229/230), Knorr (1978a, 1986, p. 230), Jones (1986a, pp. 596–598).

1. Protasis/Ekthesis
Start with a square ABCD,[1] circular arc BD (K1 in what follows), and quadratrix BT. arc BD:BC = BC:CT. (CT is the third proportional to arc BD and BC).

2. Apodeixis (by double reductio: "exhaustion"[2])
If not, then either arc BD:BC = BC:CK, CK > CT
or arc BD:BC = BC:CK, CK < CT
2.1 Assume arc BD:BC = BC:CK, CK > CT
2.1.1. Auxiliary construction
circle THKZ, center C (K2),
perpendicular HL, draw CHE
2.1.2 arc BD:BC = BC:CK = CD:CK [assumption]
CD:CK = arc BD:arc ZK [see argument *]
* argument for this (not in *Coll.* IV): similar arcs in the ratio of the radii (or diameters)[3]

$K1:K2 = CB^2:CK^2 = CD^2:CK^2$	[XII, 2]
$K1:K2 = (U1 \times CD):(U2 \times CK)$	[*Circ. mens.* I]
$CD^2:CK^2 = (U1 \times CD):(U2 \times CK)$	
$\Rightarrow CD^2:(U1 \times CD) = CK^2:(U2 \times CK)$	[V, 16]
$\Rightarrow CD:U1 = CK:U2$	[VI, 1]
$\Rightarrow CD:CK = U1:U2$	[V, 16]
$\Rightarrow$ CD:CK = arc BD:arc ZK	[V, 15]*
Thus, BC = arc ZK	[V, 9]
2.1.3 arc BD:arc ED = BC:HL	[*symptoma*]
= arc ZK:arc HK	[equal parts of quadrants]
$\Rightarrow$ HL = arc HK	[V, 9]
This is impossible.	

2.2 Assume, then, that arc BD:BC = BC:CK, CK < CT
2.2.1 Auxiliary construction:
circle ZMK, center C;
perpendicular KH, draw CHE
2.2.2 As in 2.1.1, we see: arc BD:BC = CD:CK,

[1] The labeling of corner points for the square in the starting configuration is now counterclockwise, as opposed to the original *genesis*. Perhaps this is an indication that the author of Prop. 26 is different from the source for the *genesis* and *symptoma*.

[2] On Prop. 26 see also Heath (1921, I, pp. 226–229) and Knorr (1986, pp. 226–230). Knorr's account contains some interesting speculative remarks on the study of tangents, subtangents etc.

[3] The theorem that circumferences have to one another the ratio of the respective diameters is used repeatedly in *Coll.* IV, cf. Props. 26, 30, 36, 39, and 40. A proof is given by Pappus in *Coll.* V, 11 and VIII, 22.

and CD:CK = arc BD:arc ZK,
⇒ BC = arc ZK
[use an argument analogous to argument *].
2.2.3 As in 2.1.3, we see that
arc BD:arc ED = BC:HL [*symptoma*]
arc BD:arc ED = arc ZK:arc MK [equal parts of quadrants]
⇒ HL = arc MK [V, 9].
This is impossible.

3. Symperasma: Therefore, arc BD:BC = BC:CT must hold.

Corollary

By constructing a line a with CT:BC = BC:a, and finding 4a, one has rectified the circle. For a = arc BD.

This means that the quadratrix has a rectification property, which can be derived from its *symptoma*. Further results, directly from the *symptoma*, or from the rectification property, can be found in Props. 35–41. They are much less spectacular than this one here.

5.4.5 Prop. 27: Squaring the Circle

After rectifying the circle, one can apply Archimedes, *Circ. mens.* I, and construct a triangle that has the same area as the given circle: base 4a, with a as in Prop. 26, appendix, height BC. This triangle can then be transformed into a square via II, 14.

5.4.6 Prop. 28: Geometrical Analysis, Linking the Quadratrix to Loci on Surfaces Through a Cylindrical Helix

5.4.6.1 Outline of the Analysis in Prop. 28

Start with a quadrant ABC, radius BD, E on BD, perpendicular EZ, assume that EZ:arc DC is *given*.[1]

Then E lies on a uniquely determined curve.

[1] Note that this is a response to Sporus's demand after criticizing the definition of the quadratrix via motions. He had demanded that a crucial ratio be given. In Prop. 28, it is taken as *given* in the sense of geometrical analysis.

Analysis

1. Extension of the configuration

Cylinder-segment over ABC; in it, take an Apollonian helix as *given* in position L, T, I as in the figure create a garland-shaped surface, determined by the helix

2. *Resolutio*
   2.1 I lies on a uniquely determined plane
   (a plane *given* in position[1]), through BC and ZI
   (or perhaps EZ and ZI); here the *given* ratio is used;
   2.2 It also lies on the plectoid surface created by the helix
   [use the *symptoma* of the helix[2]].
   Since the helix is also *given* in position, I lies on an intersection curve of surfaces, which is also *given* in position.
   2.3 Project this line onto the plane of the original quadrant.
   By construction, E will lie on this uniquely determined line.
3. Specification

When the *given* ratio EZ:arc DC = AB:arc AC, this line will be the quadratrix.

5.4.6.2 Intersection Plane in Step 2.1: Through EZ or BC?

Pappus' description is not sufficiently precise. In addition, there are several illegible spots in the main manuscripts for this part of the text, and they were already there when the minor manuscripts were copied. With Knorr, I favor the reading according to which the intersection plane is the one through BC and ZI, for it is obviously *given*, i.e., constructible, at this stage of the analysis (assuming that one has the helix). BC is *given* in the starting configuration, and the inclination of the sought plane toward the underlying plane is determined by the *given* ratio EZ:EI. The drawback is that with this intersection plane, the endpoint Z for the intersection curve in space is not uniquely determined. Neither will the endpoint of the resulting special case quadratrix be. If one opts for the plane through EZ and ZI, as Hultsch, and apparently Treweek did, one has to assume that EI and EZ are *given* in position. It is not clear, at this stage of the analysis, that they are.

Ver Eecke assumed that the segments ZE and EI are *given*, because they go through *given* points. One might object that if Z and E were *given*, there would be no need for further argumentation at this point. It is unclear how the points can be seen to be *given* at this stage. Ver Eecke also assumed that the intersection plane on which I lies goes through LT. One might find this objectionable, too.

Even if we cannot decide with certainty which plane is used in the analysis in Prop. 28, the main thrust of the argument is clear: it provides a conceptual connection

[1] My reading of Prop. 28 differs considerably from the one given in Ver Eecke (1933b, p. 199, #2); it is compatible, however, with the discussion in Knorr (1986); compare also the following notes on the crucial intermediate step 2.1.

[2] On the helix and its *symptoma* cf. *Procl in Eucl.* 105, 271 Friedlein, Knorr (1986, p. 295/296). The ratio of height and rotation angle is a constant, i.e. *given* with the curve.

between the quadratrix of Dinostratus and a locus created on a curved surface in space, in dependence from the Apollonian helix, and that is its purpose.[1]

5.4.7 Prop. 29: Geometrical Analysis, Linking the Quadratrix to Loci on Surfaces with Spiral

Start with a circular sector (not necessarily a quadrant) ABC, *given* in position, a radius BD, point E on it, and a perpendicular EZ, where EZ:arc DC is *given*, and EZ:arc DC = AB:arcAC (spiral-creating ratio). Assume that a spiral BHC is inscribed in the sector ABC.

Then E lies on a uniquely determined line.

5.4.7.1 Outline of the Analysis in Prop. 29

1. Extension of configuration
Cylindroid over spiral, height BH;
BH = EZ [construction],
EZ:arc DC = AB:arc AC = BH:arc DC [*symptoma*]
right cone, vertex B, generating line at an angle of $\pi/4$ with respect to the underlying plane
2. *Resolutio*
 2.1 Analytical determination of a locus for K
 K on HK, perpendicular to the plane, KH = BH
 HK is *given* in position
 K lies on the cylindroid surface,
 and on the surface of the cone
 ⇒ K on the intersection line created by those two surfaces:
 a conic spiral that is *given* in position.[2]

[1] I agree with Ver Eecke's summarizing statement: "En exposant ce premier mode de construction géométrique de la quadratrice au moyen des Lieux à la Surface, la proposition démontre donc, sans l'énoncer explicitement, une propriété remarquable de la surface de la vis a filet carré à axe vertical, a savoir que, si l'on coupe une surface hélicoide rampante ($y = x$ tang $(2\pi\ z/h)$) par un plan passant par une de ses génératrices rectilignes ($z = my$) [I opted for BC, Ver Eecke for LT], et si l'on projette orthogonalement, sur un plan perpendiculaire à l'axe de cette surface la courbe détérmineée comme section, on obtient une quadratrice de Dinostrate" (Ver Eecke 1933a, p. 199, #4).

[2] Note that the conic spiral used in Prop. 29 is not automatically accepted, as the helix in Prop. 28, and the spiral in Prop. 29 were. It must be reduced to the spiral in order to be revealed as *given* in position. This could be an indication that the Archimedean spiral and the Apollonian helix were viewed as privileged basic curves for the analytical determination of other motion curves by Pappus (cf. Molland 1976). If so: was this the case just for Pappus, or: more generally? Were these curves perhaps seen as basic for the *symptoma*-definition of higher curves, as Prop. 29, but also Prop. 28 seem to suggest? In the meta-theoretical passage, Pappus significantly speaks of quadratrices and spirals as exemplary curves for the third kind. Recall also Apollonius' claim on his helix as a basic curve, on a par with circle and straight line, reported in Proclus on authority of Geminus (*Procl. in Eucl.* 251 Friedlein). The issue cannot be pursued here.

 2.2. Analytical determination of a locus for I and E
 2.2.1 Extension of configuration, second part:
 analogous to the "garland" in Prop. 28, create a plectoid surface, derivable from the original spiral; use BL, and the conic spiral; both are *given* in position: LI moves along the spiral and BL, parallel to the underlying plane, creating a twisted surface in space that is *given* in position.
 2.2.2 I lies on that surface.
 2.2.3 I also lies on a uniquely determined plane [through BC and ZI; use the *symptoma* of the spiral].
 ⇒ I lies on the intersection curve created by those surfaces.
 2.2.4 Project this curve onto the underlying plane.
 By construction, E lies on this projection, on a uniquely determined line.
3. Specification:
When the sector ABC posited in this analysis is a quadrant, this line is the quadratrix.

5.4.7.2 Lines, Planes, and Surfaces in Prop. 29

Whereas Prop. 28 used a cylindrical helix from the start, Prop. 29 starts with a plane curve, the spiral, and constructs a curve in space from it as a first step: a conical spiral. Most commentator agree that the conical spiral is created by erecting a cylindroid over the given spiral and intersecting it with a right cone with axis BL, inclined at 45° toward the underlying plane. The point K lies on it. This much seems uncontroversial, and for this reason I have used a diagram for Prop. 29 that shows the cylindroid surface and the point K.

Different interpretations have been offered for the second part of the construction in Prop. 29. The reading offered here is minimalist, and modeled on Prop. 28. One draws the parallel LKI to BE, leaving the exact location of I open, i.e., reserving the possibility to extend KI if needed. The generator BZ with flexible endpoint, adjusted between BL and the conical spiral, creates a "plectoid," garland-shaped surface. It is intersected with the plane through BC and ZI, analogous to Prop. 28, and projected orthogonally onto the plane. This reading is only tentative. Its advantage over some other ones is that they all assume that LKI is extended to the circumference, and that the same cylindrical helix, and the same garland as in Prop. 28 is created. The text of Prop. 29 does, however, not mention the helix and seems to propose the analysis in terms of the plane spiral as an alternative to the one using a helix. It is not to be excluded that the original author of the argument in Prop. 29 did intend to show, with his analysis, how the plane spiral, the conical spiral, the cylindrical helix, and the quadratrix are all connected. The text as reported by Pappus does not explicitly say as much, though. Therefore, I opted for the minimalist reading (and accordingly, a very reduced diagram). If one accepts the presence of the helix, perhaps a reading along the lines of Commandino is the most straightforward one. Commandino does assume a cylinder in addition to the spiral-induced, cylindroid, extension of LKI to H on the cylinder surface, creation of a helix in dependence from the conical helix, with a garland-shaped surface for I. His diagram (Co p. 91) shows all these features. Commandino then assumes the creation of an intersection curve in space, and orthogonal projection onto the underlying plane as in Prop. 28. There are some problems with his reading in detail, for which see the translation. Hultsch ad locum refers to Chasles and Bretschneider, and does not offer an interpretation. His diagram is also minimal. For Ver Eecke's reading see Ver Eecke (1933a, p. 200f). Knorr (1989, p. 166f) offers an explorative interpretation of the material in Prop. 29, drawing a connection to Archimedes's study of tangent problems on the plane spiral. It is very interesting in itself; I am somewhat diffident, however, that it works well as an explanation of Prop. 29 as given in Pappus' text. Therefore, I have restricted my presentation of the content of Prop. 29 to the information as given in the text for the most part. For further clarifications and alternatives, the reader is referred to the literature mentioned above.

5.4.8 Additional Comments on Props. 28 and 29

5.4.8.1 Loci on Surfaces

As noted before,[1] Props. 28 and 29 are our only explicit sources on analysis of loci on surfaces. This means that observations drawn from them provide only limited knowledge of the discipline for which they are an example. There is a danger of over-interpretation, because we lack a context to check our reading against. The following observations on Props. 28 and 29 may nevertheless capture some representative features for this kind of mathematical approach.

1. The dominant method of investigation, and the method for determining the basic objects of study, is geometrical analysis in the technical sense.
2. Certain spiral-type curves have a privileged role, others are determined relative to them.
3. We operate with surfaces in space, created by rotation, by a motion that combines a linear progression and a rotational motion in synchrony (twisted surfaces, controlled "motions"), or by establishing cylindroid surfaces over a plane figure, and intersecting them with each other, and with planes.
4. The created curves in space are in the end projected onto the plane.
5. Because we are using analysis, the result is not a constructive solution, or a constructive *genesis* of the curve. This is also not intended. The content really is a mathematical analysis of the *genesis*, establishing unique determinateness for the "target curve" inside a configuration.

[1] For bibliographical references, see the literature given at the beginning of the chapter on Props. 26–29.

A Potential Context for the Analysis of Surface Loci: Analysis of loci and Conic Sections

Consider the parallel between items 3 and 4 above and Archytas' solution for the cube duplication with Eudoxus' procedure for his curve devised for cube duplication.[1] According to Zeuthen (1886, pp. 460–461), this procedure was taken over by Menaechmus as a model for the conic sections, viewed as analytically determined loci (cf. item 1.), as plane *symptoma*-curves.[2] Even after the conics were discovered to be sections of cones, and their definition was in terms of this *genesis*, the actual handling continued to be focused on the *symptoma*-characterization. Consideration of those aspects of the Apollonian treatment of conics that might be viewed as analogous to *symptoma*-mathematics – and there are quite a few examples (see again also Zeuthen 1886 passim) – might help to reconstruct a context for the *symptoma*-mathematics of the third kind, by studying the analogue in *symptoma*-mathematics of the second kind. Perhaps even the reduction of the conic sections as plane curves to the intersection of a plane and a surface in space (i.e., the surface of a cone) could be seen as somewhat of a model for the reductions we see in Props. 28 and 29. In addition, the analytical Euclidean work on loci on surfaces, on which Pappus comments in *Coll.* VII, and which is based on related work by Aristaeus, might be considered.[3] Perhaps the outlines of a context for *symptoma*-mathematics become visible here. The issue is worth exploring. A decisive difference, even if parallels can be found and brought to bear, would be the fact that conics can be viewed as essentially defined, although *symptomatically* handled; the higher curves cannot.[4]

Use of Analysis for the "Definition," or Determination of Curves

Without drawing far-reaching conclusions from our scarce evidence in Props. 28 and 29, one thing can nevertheless be said, and it has been somewhat overlooked in secondary literature on the propositions. The propositions have a clearly analytical character, with analysis taken in the full technical sense of the word. And it seems plausible to assume that this feature would have been typical of the geometry of the third kind. Specifically, geometrical analysis (*resolutio*) is used here, not to (only)

[1] Compare the remarks on cube duplication and conics in the commentary on Props. 23–25. For a hypothetical reconstruction of Eudoxus' curve cf. Tannery (1912, I, pp. 53–61). It seems plausible to assume that Eudoxus projected the space curves, created in Archytas' solution, onto the plane, creating a curve with which he could solve the cube duplication. It would have to be defined by deriving the characterizing properties from the properties inherent in the space curves. For the *symptoma* of the helix in Proclus cf. pp. 105 and 271 Friedlein; cf. also Knorr (1986, pp. 295–296).

[2] Cf. Knorr (1986, pp. 50–66, 112).

[3] *Coll.* VII, pp. 1004–1014 Hu (Jones 1986a, pp. 363–371, see also pp. 503–507, 591–599).

[4] Cf. Zeuthen (1886, pp. 459 ff.), Knorr (1986, pp. 61–66, 112) on the combination of essential, genetic definition, and operation with the *symptoma* in the theory of conic sections.

solve problems but to "mathematize" motion curves as *symptoma*-curves, by reducing them to properties of other curves that are taken as *given.*

Given in 28:
Per hypothesis: sector ABC, quadrant (as in quadratrix), radius BD with perpendicular; EZ, and ratio EZ:arc DC (this ratio is not necessarily the one used in the quadratrix). An Apollonian helix with a *given* progression ratio for angle:height (connected to EZ:arc DC).

Entailed: each such configuration determines, i.e., turns into a *given*, a certain unique projection curve, in direct dependence from the ratio that is embodied in the helix: a quadratrix-like curve. We get a family of curves. The quadratrix is the one where the given ratio is the same as AB:arc BC.

Given in 29:
Per hypothesis: sector ABC (not necessarily a quadrant, unlike quadratrix), radius BD, perpendicular EZ with ratio EZ:arc DC = BA:arc AC (ratio as in quadratrix) an inscribed spiral (embodies the ratio BA:arc AC)
Entailed:
(i) A conical spiral, as an intersection curve in space.
(ii) An intersection curve between two curved surfaces in space.
(iii) A certain unique projection curve, in direct dependence from the ratio embodied in the spiral: a quadratrix-like curve. We get, again, a (different) family of curves; the quadratrix is the one where the given sector is a quadrant.

The analysis in Props. 28 and 29 is restricted to the *resolutio*-phase: the phase where that which we need or want to establish is shown to be *given*, if certain other features (theorems, prior results, etc.) are posited. The arguments show that the curves in question are uniquely determined in a hypothetical sense: We cannot derive them from essential properties rooted in the archai and the principle objects of our discipline, but the properties we focus on in mathematical argumentation can be put in an exact, conceptualizable relation to properties of other entities in a specific spatial configuration (ultimately the *symptoma* of a privileged curve). The latter we just assume and posit – much like we posit the straight line and circle. This much one can assert. We will have to leave it undecided, because that is what Pappus does as well, whether this determination "saves" the curves completely, so that *symptoma*-curves were taken to be just as solidly defined as the archai of the plane and solid kind, even though the *symptoma*-approach operationalizes, lets the curve itself disappear and replaces it by a kind of relation/equation. We will not try to determine, at this point, what it means that Pappus asserts and supports the fully mathematical character of arguments about the curves, but is hesitant about the status of the curves themselves (cf. also meta-theoretical passage, where a similar ambivalence shows).[1]

[1] See the excursus below for some speculative remarks in this regard.

What does Pappus achieve with Props. 28 and 29? The reading suggested here is a sympathetic one. Pappus does not achieve, and does not believe he has achieved, the quadrature of the circle. He has not "saved" the *genesis* of the quadratrix in the sense that the curve can now be constructed geometrically, and he does not claim to have "solved" the problem. He succeeds in partially circumventing Sporus' objections, i.e., he interprets the demand that the crucial ratio be given before the curve can be accepted by showing in what sense, and to what degree, the ratio can be seen, via geometrical analysis, to be *given* in the technical sense of the word. He gets an analytical characterization, not a constructive definition. And he is explicit about that. Even so, the analytical determination has achieved something. Its effect is that the quadratrix, although not constructible, can be investigated geometrically, without conceptual inconsistencies, qua locus curve for a certain *symptoma*. It is well-defined, uniquely determined. The geometry on it is true geometry, geometry of the third kind. Its results are geometrically demonstrable properties of the curve as *symptoma*-locus. As long as we only had the genetic definition, which was conceptually inconsistent, such geometry did not have a satisfactory basis.

Even so, the issue of the quadratrix's foundation is not completely settled. As noted above, we cannot be sure just how solid the analytical basis is. In the meta-theoretical passage, where Pappus will, once again, classify this kind of mathematics as legitimate mathematics, alongside plane and solid mathematics, he also does say that the curves have a somewhat forced *genesis*, and he shows a certain hesitancy with respect to the third kind of mathematics. Also, the analytical approach leaves a gap: *symptoma* – analysis cannot guarantee that a more elementary construction is impossible for a curve thus characterized (e.g., that in certain specific cases, it might reduce to a locus of the second kind).

Most interpreters so far have not given Pappus a sympathetic hearing. One basic error, which is rather pervasive, is that they read Pappus' statement that he will provide a geometrical analysis (analuesthai) as actually saying that he claims to "solve" the problem (the quadrature) geometrically (equivalent to luesthai). As has been pointed out in the notes to the translation, this is a serious misunderstanding, for Pappus does indeed provide an analysis for the *genesis* of the curve, and he does not provide a solution of the problem. Jones (1986a, p. 598), e.g., seems to believe that Pappus is trying to give a construction of the quadratrix and remarks that, as constructions, they do not meet Sporus' objections. Similar attributions of confusion to Pappus can be found in Knorr (1978a, 1986). Knorr also offers, however, the consideration that Pappus may, after all, have tried not to meet Sporus' objections, but to circumvent them. In this respect, his reading concurs with mine.

5.4.9 Excursus: Speculative Remarks on the Potential of Analysis-Based *Symptoma*-Characterization of Higher Curves

The decisive difference between using the circle mathematically by focusing on its *symptoma*, and using the helix and other curves solely accessible through their *symptoma* is somewhat like this: We think (perhaps) we know what the circle is, essentially, and the properties we use in mathematics are seen as properties of that object, for which we can posit some kind of epistemological or ontological priority. Of the Apollonian helix we do not have such a direct grasp. It has to be constructed in thought as the thing which has the decisive property. The helix itself disappears, as it were, behind its *symptoma* in a way the circle does not. So what is the epistemological, or ontological, grounding of such curves, when they are viewed exclusively as loci for a *symptoma*?

In view of the complete absence of statements from ancient mathematicians on this question, and the deplorable lack of evidence on their actual practice in this area, it is perhaps fruitless to try and establish what the commonplace opinion among them would have been on that question. The following, speculative remarks should be taken as an elucidation of the potential impact of this question, its horizon of potential for future developments in the intellectual history of mathematics. Specifically, I have the sixteenth and seventeenth century readers in mind, and "anchor points" for the routes they took to transform mathematical investigations toward algebraization on the one hand, and infinitesimal calculus on the other. Could the ancient mathematicians, in defiance of the essentialist view on science and explanation – whether Aristotelian or Platonist – have taken the view that circle, helix, and spiral are really equivalent, because all mathematics is *symptoma*-mathematics and does not really care about the ontological status of the objects the *symptomata* of which it studies? That the circle, e.g., is treated as a locus curve just as the helix is? That what is mathematically interesting about it, its property, can equally well be seen as stemming directly from the motion generating it? Apollonius for one argued that the helix should be placed alongside circle and straight line as a basic, unanalyzed principle in geometrical argumentation. Does this imply an anti-essentialist thrust, a turn toward making locus-properties, i.e., relations, the final objects of mathematics? Are the basic items all loci, as it were, characterized as such via "defining" relations? That would make Apollonius a forerunner of the paradigm shift toward algebra that occurred in the seventeenth century. It cannot be ruled out.[1]

If such was the case, and there was an Apollonian programme to implement a new paradigm for mathematics, one in which operationalism, and the manipulation of relations are key ideas, we would have to say that the programme did not carry the day in antiquity, and the ancient research project of *symptoma*-mathematics might have died out precisely for that reason: re-channeling into the mainstream essentialist approach. What we see in *Coll.* IV, and what the seventeenth century readers saw as well, would then be like the remnants of a large-scale re-orientation project for mathematics which was abandoned, with the remnants still bearing the traces of the revolutionary ideas behind them, of this push toward operationalizing geometry into a proto-algebraic discipline. Such an ideological clash, an unsuccessful

[1] The fact that Apollonius apparently wrote a work called "katholou pragmateia" (universal treatise, attested in Marinus (Eucl. Op. 6, p. 234 according to Jones (1986a, p. 530/531), and the fact that the remarks attested in Proclus seem to point towards an attempt at radically reorganizing the foundations of Euclid's *Elements*, do invite speculations in this direction.

frontal attack against the ruling paradigm, which in turn was backed by non-negotiable essentialist convictions and preconceptions, and which in the end prevailed, would explain why Apollonius' minor analytical works were lost, why his *Konika* were stripped of their analysis-parts, which in the original must have been dominant (Pappus groups the *Konika* with the analytical works), and recast by Eutocius in purely synthetic form, and also why no works of the authors who worked on the analysis of loci on surfaces are preserved. The essentialists in the field of epistemology/theory of science would have won the day, and forced the continuation of the old paradigm.

Such a speculation is tempting. But it is equally possible that the mathematicians, including Apollonius, went along with the essentialist views on the nature of science and explanation, or – and that is perhaps the most likely option – that they did not reflect on such questions at all and just went ahead doing their mathematics of *symptoma*-curves. After all, even in the orthodox Aristotelian paradigm, any science is entitled to positing its principles and does not have to go beyond, justifying them, so that in the end, any science can do its job while focusing in on the rigorous development of arguments about *symptomata*. On the whole, we cannot get beyond the observation that geometrical analysis in the technical sense was applied to derive a hypothetical definition, or characterization, of motion curves through their *symptoma*. This was not just a side thought, since a considerable amount of sophistication and argumentation is needed to perform this task. It must have been of some importance, and served a serious purpose. Whether the result was that these curves were then seen as on a par, epistemologically, with objects like circles, straight lines, or conics, must be left undecided. Also, the details of this kind of mathematical argumentation are at present opaque to us, and certainly were so for the sixteenth and seventeenth century readers as well. This may be part of the reason why so much effort was spent on reconstructing the analytical works, and the analytical strategies of the ancients. Still, the material presented in Pappus is suggestive toward a new perspective on what mathematics essentially is, one in which analysis and operation with relations are central. One could pick up here; in a way that needs to be explored and spelled out in more detail, Vieta, Descartes, Fermat and others did.

5.5 *Prop. 30: Area Theorem on the Spherical Spiral*

context: Archimedes on spiral lines, motion curves, quadratures.
source: lost text of Archimedes.
means: I, III, V, XII, *Sph. et Cyl.* I, 33, I, 35, I, 42.
method: synthesis; infinite inscription process, quasi-infinitesimals, limit argument.
format: *genesis*-description, *symptoma*-theorem and corollary.
reception/significance: no reception, the only related extant treatise is *Sph. et Cyl.*; the addition to Prop. 30 is the first example for a quadrature of a curved surface in space.
embedding in *Coll.* IV: motif "Archimedes": Props. 13–18, 19–22, 35–38, 42–44; Archimedes, with his "mechanical" approach "frames" the treatment of motion curves; motif "spiral lines": Props. 19, 20, 26, 29, 35–38; motif "area theorem": Prop. 21; the content of Prop. 30 is not picked up again in *Coll.* IV.

purpose: illustrate the "mechanical" path for the treatment of motion curves: *symptoma*-mathematics as meta-mechanics.
literature: Heath (1921, II, 382–385), Knorr (1978a, 59–62, 1986, 162–163), Ver Eecke (1933b, 206, #2).

Prop. 30 is the first known example for the quadrature of a curved surface in space. Methodologically, it picks up the first, "Archimedean" path for dealing with motion curves. Although no author is named for Prop. 30, the theorem is usually ascribed to Archimedes. The parallels to the argumentative style and the structure of Prop. 21, especially the use of indivisibles, and the infinite inscription process, as well as the parallel argument using two figures with parallel division processes, are very compelling indeed. The spherical spiral is created by motions in the ratio 1:4. Unlike the plane spiral inscribed in a circle, the spherical spiral described here is conceptually well-defined in its *genesis* via motions.[1] The *symptoma* is directly read off from the coordinated idealized motions.[2] The theorem on the spiral uses aspects of Archimedes' "mechanical method," namely indivisibles. Mathematics appears as meta-mechanics, where mechanics itself is already highly abstract. We do not have a context for Prop. 30. It may have been part of a larger work.[3] No applications outside *Coll.* IV are attested. The only surviving Archimedean complete monograph on the *symptomata* of a motion curve is *SL*, and its argumentative method and style differ significantly from the quasi-mechanical approach attested in Props. 21 and 30. As in the case of the analytical branch of *symptoma*-mathematics, the lack of a context makes it impossible to draw far-reaching conclusions on the status of the mathematics of the third kind "Archimedean style," which, I think, is represented in Prop. 30 (see Knorr (1986) for an interesting, if perhaps sometimes speculative, evaluation of the possible development of motion curves in the generation after Archimedes). Certainly plausible is Knorr's assumption that the "mechanical" approach was picked up and put to use for analytical (*symptoma*-) mathematics, and this assumption also agrees with Pappus' statements on the geometry of the third kind in the meta-theoretical passage, as well as with his developmental story in Props. 19–30. Knorr also points out that the approach via infinitesimals and indivisibles was not pursued further. Pappus voices no objection to the result in Prop. 30, and obviously treats it as valid.

[1] Polar coordinate description for the spherical spiral: $\rho = 1/4\omega$; compare the analogous equations for the plane spiral in Prop. 19: $\rho = 1/(2\pi)\varphi$, and for the spiral as used in *SL*: $\rho = a\ \varphi$, where a is arbitrary, but fixed. Both Prop. 30 and *SL* avoid having to take recourse to π.

[2] No instruments are involved, the verb used for the *genesis* via motions is noein. We deal with abstract motions. The verb kinein, used in 19, is absent; no application context for Prop. 30 can be envisaged. Its "mechanical" character is purely theoretical.

[3] A comparison of the argumentative means in Props. 30 and 21 shows: Prop. 30 adds *Sph. et Cyl.* to V, XII, which were already used in Prop. 21. Knorr (1978a, pp. 59–62) argues that the material in Prop. 30 belonged to the heuristic version of *Sph. et Cyl.* The connections are clearly there. I doubt, however, that they are sufficient for postulating an immediate and precise relation such as the one postulated by Knorr.

Genesis of the spherical spiral: In a hemisphere, rotate the arc TNK of a quarter-circle through the pole along the base circle (arc KLM). At the same time, let a point N travel from the pole toward the base, and assume that it completes the quarter-arc at the same time in which the rotating quarter- arc completes the full circle. The traveling point describes a spherical spiral.

Symptoma: If one draws an arbitrary quarter-arc TOL, with O on the spiral, arc TL:arc TO = circumference: arc KL.

5.5.1 Proof Protocol Prop. 30

1. Protasis/ekthesis
Assume a hemisphere with pole T, surface A, and spherical spiral TOIK (area above: ASp), a quadrant ABCD of a maximum circle (area Q), and a segment ABC (area ASg).
Then A:ASp = Q:ASg.
2. Apodeixis
   2.1 Extension of the configuration and transformation of the protasis
   Construct sector AEZC (area S); show that S = Q[1]
   The protasis has now become A:ASp = S:ASg
   2.2 Auxiliary construction
   (set-up for the "exhaustion process")
   On the hemisphere, cut off a sector LTK (area: AL),
   describe a circle on the surface through O, center T,
   cutting off the surface OTN (area A(O)),
   with a sector cut off in it by KT, KL (area A'(O));
   cut off from arc ZA the arc ZE,
   as the same part as KL is of a maximum circle,
   cutting off from S the sector EZC (area: A(E));
   in it, cut off sector BHC (area: A(B))
   2.3 Lemma for the "exhaustion process"[2]
      2.3.1 arc ZE:arc ZA = arc BC:arc AC
      2.3.2 arc TO = arc BC
      2.3.3 AL:A'O = A:AO
      A = area of circle with radius TL
      [*Sph. et Cyl.* I, 33[3]]

[1] S is 1/8 of the circle with radius CA, Q is 1/4 of the circle with radius AD, $CA^2 = 2\ AD^2$.

[2] Compare Prop. 21: inscribe a sector into the spiral; then compare sector and spiral sector on the one hand, and rotation cylinder and cone-related rotation cylinder on the other.

[3] The reference to *Sph. et Cyl.* is, of course, anachronistic. The material in Prop. 30 probably predates the treatise. Archimedes must have been aware and convinced of these theorems independently of his theoretical work. It seems plausible to assume that he found the results in the context of his pre-formal research activity, using his heuristic method.

A(O) = area of circle with radius TO

[*Sph. et Cyl.* I, 42]

$A:A(O) = TL^2:TO^2$ [XII, 2]

$\Rightarrow AL:A'(O) = TL^2:TO^2$

2.3. 4. TO = BC [III, 29]

TL = AC = EC by construction

$\Rightarrow AL:A'(O) = EC^2:BC^2$

2.3.5. $EC^2:BC^2 = A(E):A(B)$ [XII, 2; V, 15]

$\Rightarrow AL:A'(O) = A(E):A(B)$

2.4 "exhaustion from above"[1]:

Iterate the process described in 2.3, and sum up;

A: sum of all circumscribed (spherical) spiral sectors =

S: sum of all circumscribed partition-induced plane sectors

2.5 "exhaustion from below"

The analogous proposition will hold for inscription instead of circumscription.

2.6 limit process

Imagine the partitions more and more fine-grained.

The inscribed and circumscribed spherical sectors approximate the spiral surface from both sides, and the inscribed and circumscribed plane sectors approximate the segment. The same propositions will always hold. By an implicit continuity argument (a transition to infinity, or an appeal to indivisibles[2]), we infer: they still hold in the limit case, and thus: A:ASp = S:ASg = Q:ASg

Addition: Quadrature of a Spiral-Induced Surfaces on the Hemisphere

Since A = 8Q by [*Sph. et Cyl.* I, 33], we can derive

(a) For the area above the spiral: ASp = 8 ASg

(b) For the area below the spiral

A – ASp = 8Q – 8 ASg = 8(Q – ASg) = 8 triangles ABC,

and triangle ABC = $1/2(1/2d)^2 = 1/8d^2$

[1] Compare Prop. 21. There the exhaustion from "within," i.e., "below" was discussed at length, and the other case glossed.

[2] An analogous limit argument was used in Prop. 21.

II, 6 Meta-theoretical Passage

6 Meta-theoretical Passage

This passage is a locus classicus for methodology in ancient mathematics. It is perhaps the best-known passage in *Coll.* IV. A doublet can be found in *Coll.* III, and a shorter version in *Coll.* VII. There are to be three kinds of mathematics: plane, solid, linear, corresponding to three kinds of basic curves. In addition, a homogeneity requirement holds: only arguments that use means from the mathematical kind to which the problem belongs are fully valid mathematically. The passage is referred to in Descartes' *Géométrie* (Descartes 1637, p. 315, pp. 40/41 Smith/Latham). Newton also quotes it with approval, and employs it against the Cartesian program in geometry. Up until relatively recently, it was taken to be the communis opinio for mathematics throughout antiquity, and quoted or referred to in secondary literature in this way. In fact, it is, at least in this generality, only to be found in Pappus. For him, it is obviously important. He is committed to this view in the following sense: he uses it to structure his material to give a representative survey of ancient mathematics, to give a coherent methodologically oriented picture of the geometrical tradition. It is not certain, and in fact not all that relevant for the understanding of *Coll.* IV itself, whether this meta-theoretical position was shared, in this full generality, by the mathematicians. Pappus may very well be generalizing a feature to be found in Apollonius' analytical works on locus problems: separate plane problems from solid ones.[1] Still, he is well-informed, competent, and manages to tell a reasonably coherent story. It should be appreciated as a whole. An extensive discussion will not be given here (for the full text, see the translation in Part I). In the present edition, I have taken this passage quite literally, and propose a reading of the whole of *Coll.* IV in light of it. In what follows, I will comment on the two main items in the passage: the mathematical kinds, and the homogeneity criterion, and briefly indicate how the different parts of *Coll.* IV relate to remarks in the passage.

6.1 The Three Kinds of Geometry According to Pappus

There are three kinds of mathematical problems, generalized to three kinds of geometry, according to the means needed to solve the problem or demonstrate features.

[1] Cf. Jones (1986a, p. 530, 540/541), Knorr (1989, p. 34) for a similar assessment (Pappus generalizing a trend to be found in Apollonius' plane analytical works); e.g.,: "Pappus is our only explicit authority on this mathematical pigeon-holing, and he says nothing about how it developed and when. However, it is difficult not to see Apollonius' two books on *Neuses* as inspired by the constraints of method imposed on the geometer.... The only conceivable use for such a work would be as a reference useful for identifying 'plane' problems." (Jones 1986a, p. 530). On p. 530f., Jones also voices the opinion that Apollonius may have had a similar purpose in the *Plane Loci* and the *Tangencies*.