The empirical basis of equilibrium: Mach, Vailati, and the lever

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Abstract

About a century ago, Ernst Mach argued that Archimedes's deduction of the principle of the lever is invalid, since its premises contain the conclusion to be demonstrated. Subsequently, many scholars defended Archimedes, mostly on historical grounds, by raising objections to Mach's reconstruction of Archimedes's deduction. In the debate, the Italian philosopher and historian of science Giovanni Vailati stood out. Vailati responded to Mach with an analysis of Archimedes's deduction which was later quoted and praised by Mach himself. In this paper, my objective is to show that the debate can be further advanced, as Mach indicated, by reframing it in terms of the empirical vs. the logical dimensions of mechanics. In this way, I will suggest, the debate about Archimedes's deduction can be resolved in Mach's favour.

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1. Introduction: Mach's question

About a century ago, Ernst Mach argued that Archimedes's deduction of the principle of the lever is invalid, since the premises of the deduction contain the proposition to be demonstrated. The core of Mach's criticism is directed at what I call the *equilibrium-preserving assumption*.[1] It can be stated as follows.

> *Equilibrium-preserving assumption*. If weights are in equilibrium on a horizontal, weight-less beam, free to rotate around a fulcrum, then equilibrium is not disturbed if one of the weights is replaced with two weights

placed at equal distances from the point at which the original weight was placed, each of the two weights being equal to one half the original weight.

Subsequently, many scholars defended Archimedes by raising objections to Mach's historical reconstruction of Archimedes's deduction. I too consider Mach's reconstruction inadequate and misleading on grounds of historical accuracy. In fact, in the wake of the scholars who defended Archimedes, I am convinced that it is only by painstakingly piecing together the form of reasoning by which Archimedes thinks it necessary, in his deduction, to proceed in the way he does that we can deepen

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[1] Mach (1960), pp. 13–20. For Archimedes's original proof, cf. Archimedes (1913), pp. 133–137; for an English version, cf. Dijksterhuis (1987), pp. 289–290.

our understanding of the history of mechanics.[2] In what follows, I will thus leave aside the historical dimension of the debate sparked by Mach, which I consider to have been clarified, if not successfully resolved, by subsequent historians and philosophers of science. This scholarship has tended to emphasize the role of Archimedes's lost works about centres of gravity, in order to understand why and to what extent Archimedes regarded the equilibrium-preserving assumption as self-evident. Indeed, the question of the validity of the equilibrium-preserving assumption can be traced back to the so-called Renaissance 'revival of Archimedes', and in particular to the edition of Archimedes's *On plane equilibrium* edited by the Italian mathematician, Guido Ubaldo dal Monte (1545–1607).[3] However, it is also worth noting that the equilibrium-preserving assumption might have been questioned by Archimedes himself, if Wilbur Knorr is right in suggesting that an Arabic treatise on the lever by Thābit ibn Qurra (ninth century)—where Thābit purportedly gives a proof of the equilibrium-preserving assumption—is modelled on a lost work by Archimedes.[4] Hence, many studies have shown that, on historical grounds, Mach's opinion about Archimedes was inadequately informed or wrong.

However, Mach's reconstruction of Archimedes's deduction raises broader issues about mechanics, for example about the nature and role of apparently self-evident assumptions in statics and dynamics. These issues, concerning especially the principles of equilibrium—whether, for instance, they have an empirical origin, or an essentially logical, that is, self-evident character—have been overshadowed by the predominantly historical reaction to Mach's criticism of Archimedes. I call the broader issues raised by Mach the *philosophical* side of the debate.

On the philosophical side of the debate, the Italian philosopher and historian of science, Giovanni Vailati (1863–1909), stands out.[5] Vailati responded to Mach with an analysis of Archimedes's deduction which was later quoted and praised by Mach himself.[6] Vailati's counter-argument to Mach is fascinating. It seems to have led Mach, in his brief response to it, to refocus the question that had troubled him in his earlier criticism of Archimedes. Above all,

the Mach–Vailati exchange allows us to pierce deeply Mach's philosophically acute (but historically blunt) criticism of Archimedes. Mach's original question was the following: 'From the mere assumption of the equilibrium of equal weights at equal distances is derived the inverse proportionality of weight and lever-arm! How is that possible?'[7]

Vailati's counter-argument, as we shall see, interpreted Archimedes's deduction as leading to a conclusion applicable to a larger class of equilibrium-preserving transformations than that allowed by the equilibrium-preserving assumption.

Subsequently, Mach responded to Vailati by reframing his original question. 'But the aim of my whole book is to convince the reader that we cannot make up properties of nature with the help of self-evident suppositions, but that these suppositions must be taken from experience'.[8]

In this paper, my objective is to show that the philosophical side of the debate can be further advanced, as Mach intimated to Vailati, by reframing it in terms of the empirical vs. the logical (that is, self-evident, in Mach's language) dimensions of mechanics. More specifically, I will suggest, the philosophical side of the debate concerning the properties of equilibrium of the lever can be resolved in Mach's favour. Thus, I will argue that, in the case of equilibrium of the lever, the answer to Mach's question is 'Impossible!', and show that Vailati's counter-argument is untenable.

In the next Section, I will discuss equilibrium in gravity fields by means of a simple two-dimensional model of punctiform masses connected by mass-less rigid rods. This simple model suffices to show the empirical nature of the rules of equilibrium of the lever, on the grounds of which I will answer to Mach's question: 'Impossible!'. In Section 3, I will present Vailati's counter-argument to Mach, and show that it is untenable, though it has the great merit of sharpening our understanding of the empirical vs. the logical dimensions of mechanics. In Section 4, I will draw the conclusions by performing a thought experiment with a balance placed in imaginary gravity fields.

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[2] Cf. Mach (1883), Hölder (1900), pp. 63 ff., Duhem (1905–1906), Vol. 1, p. 356 (who quotes Mach but surprisingly does not comment on Mach's criticism of Archimedes), Hölder (1924), pp. 39–45, Lenzen (1932), Reimann (1936), Stein (1965), Goe (1972), Schmidt (1975), Sato (1981), Beisenherz (1981), especially pp. 451 ff., Vailati (1987), Vol. 2, pp. 101–112, 220–225, Dijksterhuis (1987), pp. 291–304, and Wilbur Knorr's résumé of the status quaestionis in Dijksterhuis (1987), pp. 435 ff. More recently, cf. Renn, Damerow, and McLaughlin (2003).

[3] With the expression 'Renaissance revival of Archimedes' scholars refer to the flurry of editions of, and commentaries on, works by Archimedes that culminated in the sixteenth century. Cf., for example, Archimedes (1544, Commandino (1565), Dal Monte (1588), pp. 55–60, and the recent studies by Bertoloni-Meli (1992), Hon and Goldstein (2005), Van Dyck (2006), and Palmieri (in press).

[4] Knorr (1982) and Jaouiche (1976).

[5] See a general introduction to Vailati, by Mauro De Zan, on the website of the *Centro Studi Giovanni Vailati* (De Zan, n.d.).

[6] Vailati (1987), Vol. 2, pp. 220–225. Vailati's paper was originally presented at the 1903 *International Congress of Historical Sciences* in Rome, and published in 1904 (cf. ibid., p. 220). Cf. Mach's praise of Vailati's paper, in Mach (1960), p. 27. Eleven letters of Vailati to Mach have been preserved, which bear witness to Valilati's admiration for the Austrian–Moravian philosopher. See Vailati (1971), pp. 111–130.

[7] Mach (1960), p. 19. Mach's question shows his inaccuracy on matters of historical reconstruction. Archimedes does not derive the inverse proportionality of the principle of the lever. On the contrary, he starts from it and derives the physical fact of the equilibrium of the lever by tacitly using the equilibrium-preserving assumption. Cf., for example, Dijksterhuis (1987), pp. 289 ff.

[8] Mach (1960), p. 27.

2. Equilibrium in gravity fields

Consider a two-dimensional system of N punctiform masses attached to one another by mass-less but rigid rods (Fig. 1). Consider any one whatever of the masses, and let it be called M_P, with the subscript P = 1 ... N. Imagine forces being applied to the masses. Let vector M_{PO} be the *moment* of force F_P with respect to point O, that is, the vector product of the force acting on mass P, and the distance, PO, assumed as a vector, of mass M_P from point O (vector quantities are represented in bold type). The equilibrium of the system will be governed by two laws of the statics of rigid bodies.[9] First, the vector sum of all the forces $(\sum F_P)$ must be zero (law of force). Second, the vector sum of all the moments $(\sum M_{PO})$ must be zero for a point O (law of moment).[10] These two laws are necessary and sufficient conditions for the equilibrium of rigid bodies.

The vectors of the moments will be perpendicular to the plane of the masses, their direction depending on the conventional choice of direction for the vector distance PO (in Fig. 1, outwardly from the plane of the page, towards the reader).

Suppose now that there is a point C such that $\sum M_{PC} = 0$ and that $\sum F_P$ is not zero. The system is not in equilibrium, then. However, now imagine removing all the single forces F_P, and adding a sum total force, $\sum F_P$, applied to point C. This transformation will not alter the zero total moment with respect to C (the distance of $\sum F_P$ from C being zero). The total moment with respect to C was and remains zero. This transformation, then, does not affect the state of initial disequilibrium of the system of masses, since they are all connected by rigid rods, and we have only replaced the single forces with their sum total, without altering their total moment with respect to point C. Now, imagine adding to the system a second force $(-\sum F_P)$ (the black thick arrow in Fig. 1), equal in absolute value but opposite in direction to $\sum F_P$, and applied to point C. With the second force added there will be equilibrium. For, $(\sum F_P - \sum F_P)$ will certainly be zero, and since now the only forces acting on the frame of masses, $\sum F_P$ and $-\sum F_P$, are both applied to C, their total moment with respect to C will be zero. I imagine this point rigidly connected to the frame of masses with mass-less rods. Let this point be called the *centre of action* of forces F_P.

Now suppose the forces are being generated by a gravity field constant in strength and direction throughout the two-dimensional space. Every mass will be subject to a force proportional to the mass, and all the forces will have the same direction (Fig. 2). I will assume that a centre of action
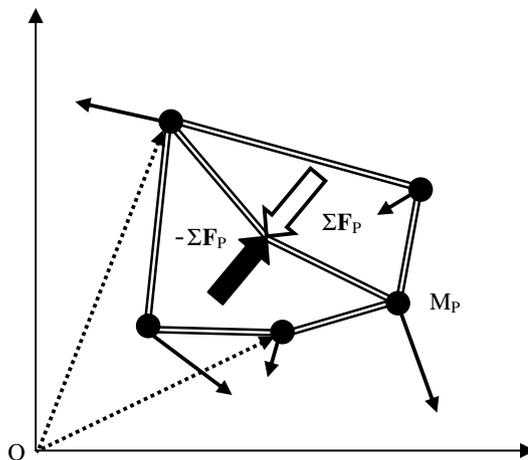

Fig. 1. The equilibrium of a two-dimensional system of punctiform masses, M_P, subject to any forces whatever, and attached to one another by mass-less but rigid rods (five masses, in this example).
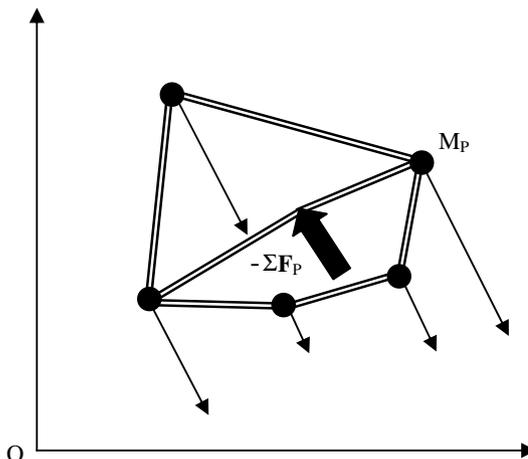

Fig. 2. The equilibrium of a two-dimensional system of punctiform masses, M_P, attached to one another by mass-less but rigid rods in a constant gravity field.

always exists, and that a force $-\sum F_P$ is being applied to it in order for the system to be in equilibrium.

Imagine $-\sum F_P$ (black thick arrow in Fig. 2) being a force equal in absolute value but opposite in direction to $\sum F_P$. Let the centre of action of F_P be called the *centre of gravity* of F_P, or, for simplicity, the centre of gravity of the system of punctiform masses. Thus the centre of gravity is simply a special instance of the centre of action of forces, obtaining when the gravity field is constant in strength and direction throughout the two-dimensional space, in which case I call the forces applied to the masses

[9] Cf., for example, the classical treatise by Paul Appell (1855–1930), in Appell (1991), pp. 149–151 (who, however, does not use the word 'law'). See Parkus (1966), on the mechanics of rigid bodies, especially pp. 28–33, on equilibrium. The treatment of the equilibrium of systems of forces acting on rigid bodies was systematized in the nineteenth century by August Möbius (1790–1868). See Möbius (1886), Ziegler (1985), pp. 29–39, and Grey (1993). In this paper, I have used 'law' in the sense of 'empirical regularity'. The laws of the statics of rigid bodies are in accord with all our experiences concerning the equilibrium of real bodies that are similar enough to ideally rigid bodies.

[10] If the vector sum of all the moments is zero for a point O, then it will be zero for all points. Cf. Appell (1991), p. 26.

weights. Hence, the centre of gravity is the centre of action of weights.

Now imagine a system of only two masses, M_{P1}, M_P (Fig. 3). I call such a system a *balance*. Also imagine a gravity field being generated by another punctiform mass fixed in space at point O. All forces will be directed towards point O. I call point O the *centre of the earth*.

Furthermore, I call any point fixed in space, around which a balance can turn (around an axis perpendicular to the page), the *fulcrum* of the balance. The fulcrum introduces a constraint into a system of masses in equilibrium. The fulcrum only allows rotation of the balance. For example, consider a balance in equilibrium under the hypothesis that, at the surface of the earth, the gravity field is constant in strength and direction, at least within some small portion of the two-dimensional space. The forces applied to the masses of the balance, then, become what I have called weights. If the centre of gravity of the weights coincides with the fulcrum of the balance, then the balance remains in equilibrium (Fig. 4). Note that both centre of action and centre of gravity are properties of systems of forces, not properties of objects such as balances.

Let us take stock. In our two-dimensional model, there is no distinction between a balance and a more complex frame of masses, or between a balance placed in a complex gravity field and a balance placed in a simple one, except that the calculations needed to determine the conditions of equilibrium will be more complicated. Equilibrium depends solely on the forces applied to the masses. From now onwards we will consider forces generated by gravity fields.

How does equilibrium change, if at all, when we allow ourselves the possibility of varying the position and orientation of a frame of masses initially in equilibrium in a gravity field in the two-dimensional space? I call the possi-

Fig. 4. A balance in equilibrium at the surface of the earth. All forces will be parallel, that is, equal in direction and, for instance, directed downwards. The forces are, in this case, what I have called weights.
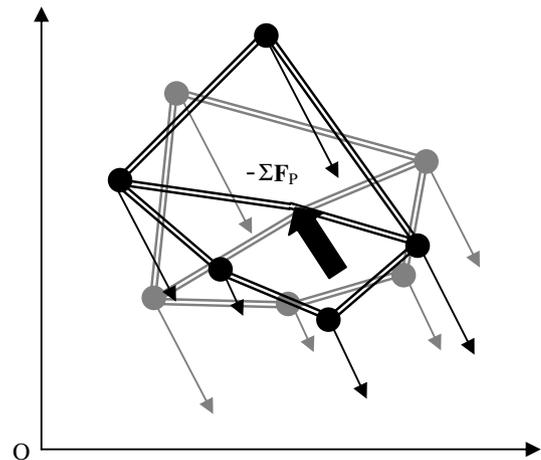
Fig. 5. The equilibrium of a two-dimensional system of punctiform masses in a constant gravity field. Is equilibrium preserved when the frame of masses is rotated? Note that the direction of the weights does not change.

bility of varying position and orientation *transport*. Is equilibrium preserved or disturbed when we allow a transport of the frame of masses? In other words, are there special circumstances under which the equilibrium of a frame of masses is invariant to transport? Clearly, the answer depends on the nature of the gravity field generating the forces.

Let us consider a frame of masses in equilibrium in a constant gravity field (Fig. 5). Translation does not affect equilibrium since the field is the same everywhere. How about rotation? Note that the direction of the weights does not change, since we assume a constant gravity field. For the same reason force $-\sum \mathbf{F}_P$ (thick black arrow) does not change in value and direction. The centre of gravity will in general change its position relative to the chosen coordinate system, but it will not change its position relative to
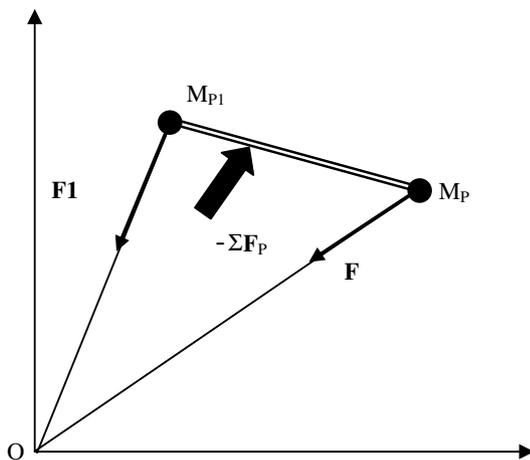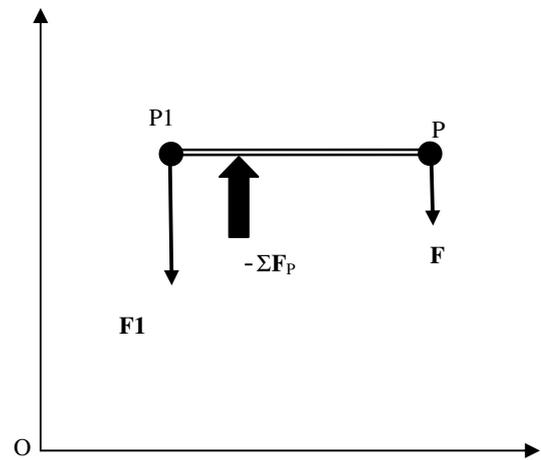

Fig. 3. A balance in equilibrium in a non-constant gravity field. The gravity field is generated by another point mass (not shown in the diagram) fixed in space at point O. All forces will be directed towards point O and be different according to mass, position of the mass, and distance from the centre.
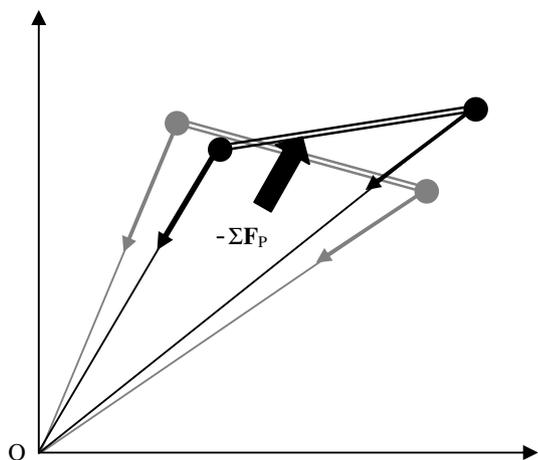
Fig. 6. The equilibrium of a balance in a non-constant gravity field. The gravity field is generated by another punctiform mass fixed in space at point O. All forces will still be directed towards point O, but, in general, the centre of action will change its position relative to the masses of the balance.
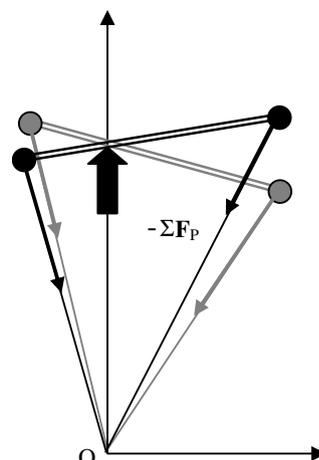

Fig. 7. The equilibrium of a balance in a non-constant gravity field. The case of a balance initially placed in equilibrium that is free to rotate around a fixed fulcrum. Let us recall that 'rotation' is around an axis perpendicular to the page since our model is two-dimensional.

the masses of the rigid frame.[11] In the example (Fig. 5), the frame of masses rotates around the centre of gravity. Thus, the position of $-\sum \mathbf{F}_P$ (thick black arrow) after rotation is the same as before rotation. Rotation, therefore, and obviously translation, do not affect equilibrium in a constant gravity field. Thus, we can assert that in a constant gravity field equilibrium will be invariant to transport.

In consequence, in the case of a balance on the surface of the earth, under the hypothesis that the gravity field is constant in strength and direction (Fig. 4), equilibrium is invariant to rotation of the balance around the fulcrum, and more generally to transport of the balance. However, it is also obvious that the equilibrium of a balance in a non-constant gravity field, such as, for example, the field generated by a punctiform mass at the centre of the earth, is much more complex.

Consider first the general case of the balance rotating and translating (Fig. 6). Since the forces applied to the two masses change because of the changing distance from the centre of the earth and because of the changing orientation of the lines directed towards the centre of the earth, the centre of action will, in general, change its position relative to the masses of the balance. Now consider the simpler case of a balance initially placed in equilibrium and free to rotate around the fulcrum (Fig. 7). What happens if the balance is rotated around the fulcrum?

Since the fulcrum is a point fixed in space the question of equilibrium can be decided by simply determining whether the total moment around the fulcrum remains zero. Now, since the force applied to the mass on the right decreases, because its distance from the centre increases, and since the vice versa occurs on the left, a net moment will appear
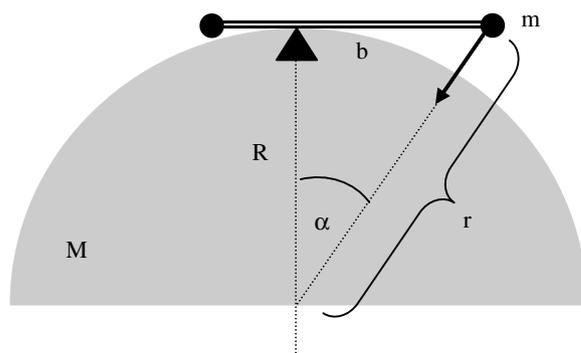

Fig. 8. The condition of equilibrium of a non-Archimedean balance in a non-constant gravity field generated by a punctiform mass placed at the centre of the earth. The balance is located on the surface of the earth and is perpendicular to the radius passing through the fulcrum.

tending to make the balance rotate counter-clockwise. In consequence, the balance will no longer remain in equilibrium. The equilibrium of the balance is, in other words, not invariant to rotation around the fulcrum. For equilibrium to be preserved the fulcrum would have to be relocated further from its original position towards the left.

What is the general condition of equilibrium of a balance in a non-constant gravity field? I will call such a balance a *non-Archimedean balance*. Let us assume a gravity field generated by a punctiform mass placed at the centre of the earth. Let us consider a balance placed on the surface of the earth. We shall determine the condition of equilibrium of the non-Archimedean balance, and show the special circumstances under which it reduces to the well known condition of equilibrium that was first established by Archimedes (Fig. 8).

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[11] It can be proven that, because of the particularly simple nature of the constant gravity field, the centre of gravity remains indeed in the same position relative to the masses of the frame. This proof can be found in the relevant literature. Cf., for example, Mach (1960), pp. 21–24.
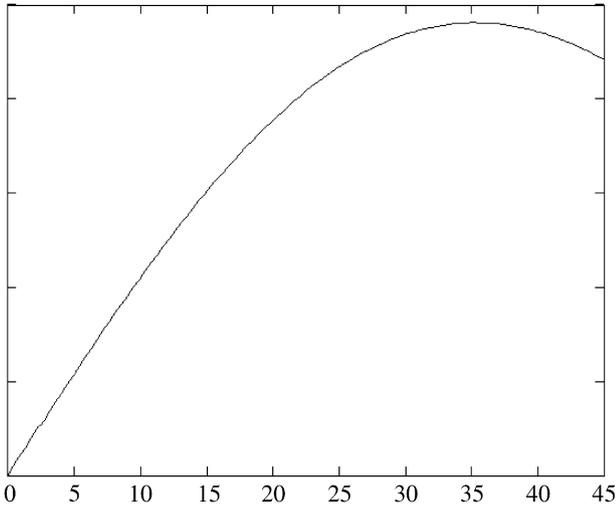
Fig. 9. The moment of around the fulcrum of mass *m* on a non-Archimedean balance is not proportional to the distance of *m* from the fulcrum. On the horizontal axis the angle α is represented varying in degrees from 0° to 45°. On the vertical axis the value of function $\mu(\alpha)$ is represented. A maximum is reached at approximately 35°. The EARTH Diameter used for this calculation is 12,7563 KM.

The moment μ around the fulcrum of a mass *m*, placed at distance *b* from the fulcrum, is given by the following function of angle α,

$$\mu(\alpha) = \frac{m \cdot M \cdot G}{R} \cdot \sin(\alpha) \cdot \cos(\alpha)^2 \tag{1}$$

where *M* is the mass of the earth, *R* is the radius of the earth, and *G* is the constant of universal gravitation (see Appendix). The moment is not proportional to the distance from the fulcrum, as the following diagram illustrates for a particular case (Fig. 9).

Let us now consider two masses, and call m_1 that on the right and m_2 that on the left. To simplify things, let us imagine that the fulcrum (the small black triangle supporting the balance, cf. Fig. 8) will always react to the sum total of the applied forces with a reaction force equal to $-\sum \mathbf{F}_P$, so that the condition of the law of force will always be satisfied. The law of moment, then, tells us that for equilibrium to obtain, the following relation, which simply expresses the ratio of the moments around the fulcrum, as given by (1), must hold true; that is,

$$\frac{m_1}{m_2} = \frac{\sin(\alpha_2) \cdot \cos(\alpha_2)^2}{\sin(\alpha_1) \cdot \cos(\alpha_1)^2} \tag{2}$$

Let us call (2) the *law of the non-Archimedean balance*. It obtains on the surface of the earth when the gravity field generated by the whole mass of the earth, supposed to be concentrated in the centre, is properly taken into consideration. Under which circumstances does the *law of the non-Archimedean balance* reduce to the well known condition of equilibrium of the Archimedean balance? If we imagine the angles tending to zero, that is, the non-Archimedean balance having arms very small compared with the radius of

the earth, then, in the limit, the arms of the balance (let us call them b_1, b_2) will become proportional to the angles α_1, α_2, since in the limit (2) will yield

$$\frac{m_1}{m_2} = \frac{\alpha_2}{\alpha_1} = \frac{b_2}{b_1} \tag{3}$$

which is the familiar condition of equilibrium of the Archimedean balance. As expected, expression (3) tells us that weights are inversely as the arms of the balance. So, for very small balances (compared to the radius of the earth) the law of the non-Archimedean balance reduces to the well known Archimedean condition of equilibrium. Indeed, for very small balances (compared to the radius of the earth) the gravity field in proximity of the balance can be thought of as being constant in absolute value and direction.

If we now call a rotation plus a translation a *rigid transport*, we can say that so far we have investigated conditions that render equilibrium invariant to rigid transport. But we can imagine more complex transformations than *rigid transport*— that is, transformations which do not preserve the geometrical configuration of the rigid frame of masses. In particular, we can imagine relocating some of the masses of the system under investigation, or even splitting some of the masses. The latter possibility is most relevant since it is precisely the case that originated so much controversy about Archimedes.

Let us first consider the balance under the hypothesis that at the surface of the earth the gravity field is constant in strength and direction throughout at least some small portion of the two-dimensional space (Fig. 10).

Imagine splitting one of the two masses, say, P, in two equal masses and relocating them, by placing one at a certain distance further away from the fulcrum, and the other closer to the fulcrum by the same distance. For equilibrium to be preserved the condition of the law of moment must be satisfied (that the condition of the law of force is satisfied is obvious). We are at liberty to chose any point for the cal-
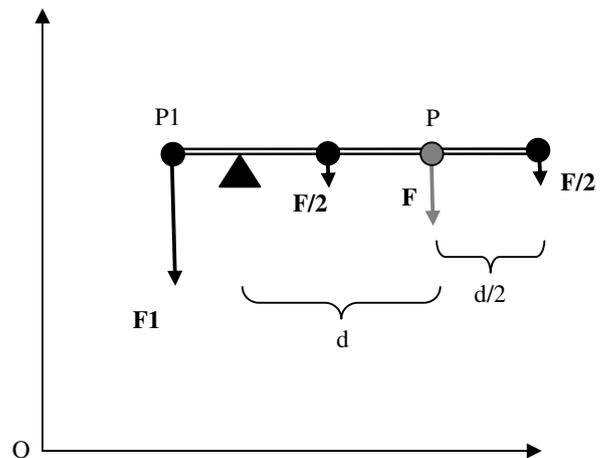

Fig. 10. The equilibrium of a balance at the surface of the earth. Relocating two masses in a way that is symmetrical with respect to an axis parallel to the direction of the gravity field.
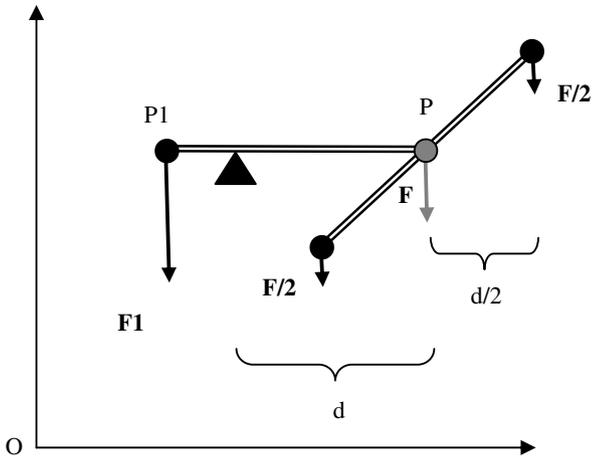
Fig. 11. The equilibrium of a balance at the surface of the earth. Relocating two masses in another way that is symmetrical with respect to an axis parallel to the direction of the gravity field.
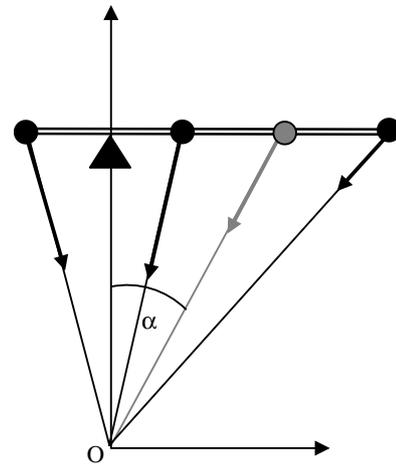

Fig. 12. The equilibrium of a balance in a non-constant gravity field. For equilibrium to be preserved both the condition of the law of moment and the condition of the law of force must be satisfied.

culation of the moments. Let us assume the fulcrum. The moment of **F**, that is, before the relocation, is **F**d. The total moment due to the two split masses is $\frac{\mathbf{F}}{2} \cdot \frac{d}{2} + \frac{\mathbf{F}}{2} \cdot \frac{3}{2}d = \mathbf{F}d$, that is, the same as before relocation. Equilibrium is therefore preserved. Thus, the equilibrium of a balance in a constant gravity field is invariant to splitting, and to any symmetrical relocation with respect to an axis parallel to the direction of the gravity field. That the latter generalization is true can be suggested informally (cf. Fig. 11).

Now, imagine repeating the same procedure for a balance in equilibrium in a non-constant gravity field, such as the field really present at the surface of the earth. Imagine again splitting one of the two masses of the balance in two equal masses and relocating them by placing one at a certain distance further away from the fulcrum, and the other one closer to the fulcrum by the same distance. Note that, since the gravity field is not constant, the vertical axis can no longer be said to be parallel to the constant direction of the field. Above all, the calculation of the condition of equilibrium is more complicated, as we shall see presently (Fig. 12). For equilibrium to be preserved both the condition of the law of force and the condition of the law of moment must be satisfied.

To simplify things, let us assume again that the fulcrum will always react to the sum total of the applied forces with a force equal to $(-\sum \mathbf{F}_P)$, so that the condition of the law of force will always be satisfied. For equilibrium to be preserved, then, the sole condition of the law of moment need be satisfied. To determine whether it can be satisfied, we need to specify the gravity field. Let us assume a gravity field generated by a punctiform mass placed at the centre of the earth. As we have seen, this is the case of the non-Archimedean balance. The moment is not proportional to the distance from the fulcrum. Since we already know $\mu(\alpha)$, we can do the calculation by adding together the expressions of the two moments $\mu(\alpha)$ for the two split masses (see Appendix). The following diagram shows the ratio of the combined moment of two split masses to the

moment of the single mass, as a function of the distance, from the single mass before splitting, at which the two masses are relocated, and for one particular case of a mass initially located, before splitting, at a distance from the fulcrum corresponding to an angle $\alpha = 20°$ (Fig. 13).

The ratio starts from one, at relocation distance equal to zero, and diminishes gradually as the split masses are relocated at greater and greater distances from the point where the single mass was initially located. This means that the moment of the split masses becomes smaller and smaller than the moment of the single mass before splitting. The equilibrium of the non-Archimedean balance, therefore, will not be preserved. The non-Archimedean balance tips to the other side of the mass that has been split into two masses.
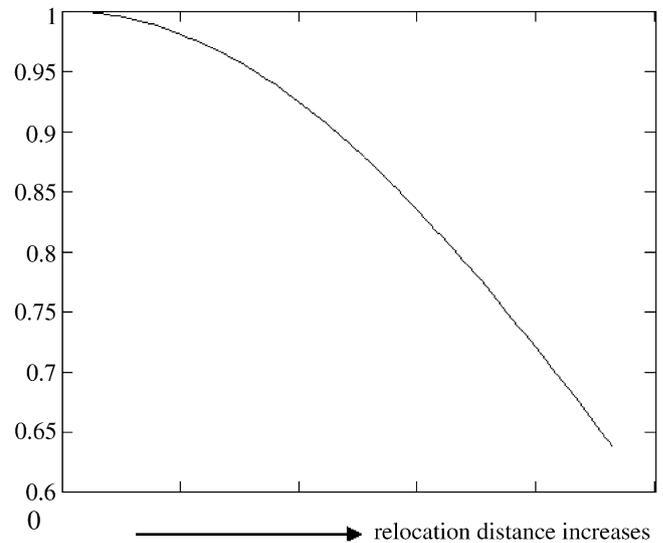

Fig. 13. The ratio of the moment of two split masses to the moment of a single mass, when the two masses are relocated at equal distances starting from zero. The diagram represents the particular case of a mass initially located at a distance from the fulcrum corresponding to an angle $\alpha = 20°$.

To sum up, equilibrium of masses attached to one another with rigid rods in gravity fields is determined by the complex relation among the forces generated on the masses by the field. For equilibrium to obtain, the conditions of the law of force and the law of moment must be satisfied. The particular nature of the gravity field determines the invariance of equilibrium to transport, or to other transformations which do not preserve the geometrical configuration of the rigid frame of masses. More specifically, a balance placed in the earth's gravity field does not satisfy the familiar Archimedean condition of equilibrium, but the *law of the non-Archimedean balance*, which only reduces to the familiar Archimedean condition in the case of a gravity field constant in absolute value and direction. The equilibrium of a non-Archimedean balance is not invariant either to rotation around the fulcrum or to relocation of split masses.

A lever on the surface of the earth is a non-Archimedean balance. I thus suggest that the equilibrium-preserving assumption cannot be justified empirically (except, obviously, as an approximation valid for practical purposes), and that equilibrium preserving displacements are not self-evident. The displacements that preserve the equilibrium of a lever are contingent on universal gravitation's giving rise to the specific form of the earth's gravity field. The answer to Mach's question, in the case of the lever, is 'Impossible!'.

3. Vailati's counter-argument to Mach

Vailati interprets Mach's criticism of Archimedes as follows. In Mach's view, Vailati says, the equilibrium-preserving assumption seems to presuppose already that equilibrium cannot be disturbed but by displacements of the weights that alter the sum of the products of the weights and their distances from the fulcrum.[12] To Mach's view, which, Vailati continues, tends to consider Archimedes's proof as circular, the following counter-argument can be opposed. The equivalence, asserted by Mach, between the equilibrium-preserving assumption and the Archimedean law of the lever is only partial. For, it is true that the displacements of weights that, according to the equilibrium-preserving assumption, are assumed not to alter equilibrium, are among those which the Archimedean law of the lever, too, asserts not to alter equilibrium. But the converse is not true, since the Archimedean law of the lever considers a larger class of displacements of weights that includes the class considered by the equilibrium-preserving

assumption. In other words, Vailati concludes, the premises of Archimedes's reasoning assert only a part of the general conclusion at which Archimedes arrives; for, by starting from premises which assume that equilibrium is not altered by the class of displacements of weights allowed by the equilibrium-preserving assumption, Archimedes demonstrates that equilibrium is not altered by the larger class of displacements of weights allowed by the Archimedean law of the lever.[13]

Vailati believes, in other words, that the Archimedean law of the lever, that is, on his understanding, the proportionality of moment to distance from the fulcrum, implies that there exists a larger class of displacements of weights allowed by the Archimedean law of the lever which do not alter equilibrium. Vailati clarifies his counter-argument with the following notation. The equilibrium-preserving assumption can be expressed as

$$p \cdot F(d) = \frac{p}{2} \cdot F(d+h) + \frac{p}{2} \cdot F(d-h) \qquad (4)$$

where h is a length whatever (providing that $d - h > 0$), p a weight, and F an otherwise unspecified function of distance d of weight p from the fulcrum. If we now consider, according to Vailati, the Archimedean law of the lever as expressed by the following equation

$$p \cdot d = p_1 \cdot d_1 + p_2 \cdot d_2 \qquad (5)$$

where p_1, p_2 are two weights in equilibrium with p, on the other side of the lever, we will then realize that (5) allows for a larger class of displacements of weights, which do not alter equilibrium, than the class allowed by the equilibrium-preserving assumption. For, the former is not restricted to the symmetrical displacements allowed by the latter. In other words, (5) states that if weights are in equilibrium on a horizontal, weight-less beam, free to rotate around a fulcrum, then the equilibrium will not be disturbed if one of the weights, p, is split into two weights whatever, p_1, p_2, and replaced by the two weights at distances d_1, d_2 from the point at which p was originally placed, under the condition that equation $p \cdot d = p_1 \cdot d_1 + p_2 \cdot d_2$ is satisfied.

In effect, if Vailati's counter-argument is correct, we must conclude that Archimedes's proof of the equilibrium of the lever succeeds in enlarging the class of phenomena to which the equilibrium-preserving assumption is applicable. Archimedes, in other words, would succeed in showing that, from what he assumes as self-evident premises, a conclusion can be derived which is valid for a vaster class of phenomena which are not self-evident, and which, as we

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[12] Vailati (1987), Vol. 2, p. 221.

[13] Ibid. Note that Vailati says that 'the premises of Archimedes's reasoning assert only a part of the general conclusion' (my translation). Obviously, premises do not assert conclusions. Rather a conclusion is derived from premises by some inference. So, I believe, Valilati's language is somewhat loose here, and the meaning of Vailati's passage is perhaps that the premises somehow contain only a portion of the conclusions. Vailati's original passage is as follows: 'la premessa, su cui si basa il ragionamento d'Archimede, asserisce solo una parte della conclusione generale a cui egli arriva partendo da essa' (ibid.). Further, Vailati uses the comparative phrase 'larger class of displacements', but he does not specify the term of comparison. However, as we can deduce from its contextual use, the sentence should be completed with the term of comparison 'than the class allowed by the equilibrium-preserving assumption'. I owe this remark to an anonymous referee, whom I wish to thank, and who noticed Valilati's problematic and elliptical phrasings.

have seen, are contingent on the nature of gravity fields. How is that possible? It is not. To see why, we need to show that (5) is in fact derivable from (4), without the need for further empirical information. Consider the following set of conditions (6), which I call the *Vailati constraint*.

$$
\begin{cases}
2 \cdot F(d) = F(d+h) + F(d-h) \\
F(0) = 0 \\
F \quad \text{real, positive, continuous, and strictly} \\
0 \leqslant h \leqslant d \quad d, \ h \ \text{real,} \\
0 \leqslant d < \infty
\end{cases}
\tag{6}
$$

As we shall see in a moment, it can be shown that the function $F(d)$ that satisfies the Vailati constraint is unique, and that it is a linear function of the form $F(d) = Kd$, where K is a positive real constant.[14]

Therefore, it is not true that the proportionality of moment to distance from the fulcrum implies a larger class of displacements that do not alter equilibrium than allowed by the equilibrium-preserving assumption. All the displacements allowed by the proportionality of moment to distance are already allowed by the equilibrium-preserving assumption, since the unique function which satisfies the Vailati constraint is $F(d) = Kd$. In other words, the only law of equilibrium of the lever compatible with the equilibrium-preserving assumption is the proportionality of moment to distance from the fulcrum. For example, the *law of the non-Archimedean balance* is not compatible with the equilibrium-preserving assumption.

I will first give an intuitive argument that the function $F(d)$ satisfying the Vailati constraint is unique and linear, and subsequently furnish a more stringent one. Imagine $F(d)$ as a curve represented in a x–y diagram (Fig. 14). Imagine $F(d)$ being approximated by small rectilinear segments conjoined together. The Vailati constraint requires that, for every point X, the value at X is the *mean value* of two values corresponding to any two points symmetrically placed with respect to X (under the restrictions (6)).

At each conjunction point the slopes of the segments will in general be different. Therefore, at a conjunction point, X, the ordinate of the curve cannot be the mean value between any two points symmetrically placed with respect to X. For example, at X, as shown in the diagram, the ordinate of the curve would not be the mean value of the values corresponding to two points A, B symmetrically placed with respect to X. If the slopes of the two segments were equal, and if points A, B were located within the interval determined by the points $X - 1$, $X + 1$, then, the ordinate of the curve would indeed be the mean.
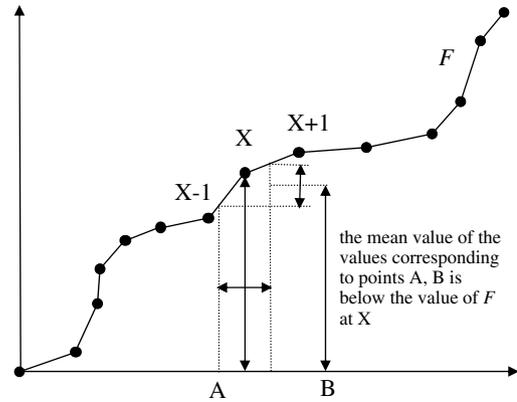

Fig. 14. An intuitive argument that $F(d)$ satisfying constraint (6) is unique and linear. Imagine $F(d)$ being approximated by small rectilinear segments. At each conjunction point the slopes of two segments will in general be different.

A more stringent argument can be given as follows. From the Vailati constraint, we have the following equality of ratios,

$$
\frac{F(d) - F(d-h)}{h} = \frac{F(d+h) - F(h)}{h}
$$

which, if the limits of the two ratios exist for h → 0, yields the equality of the derivatives at d from above and below.[15] Thus, function $F(d)$ is differentiable at all points. The argument is now a matter of simple calculus. Consider $2 \cdot F(d) = F(d+h) + F(d-h)$ and differentiate with respect to h, while keeping d fixed. We find $0 = F'(d+h) - F'(d-h)$. Since we can set d and h to be anything we like—under the restrictions (6)—we have F'(*any value*) = F'(*any other value*), which can happen if and only if F'(d) = constant, that is, F(d) = Kd, with K constant.[16]

In sum, Vailati's counter-argument is not correct. I conclude that Archimedes's proof, based on the equilibrium-preserving assumption, does not succeed in enlarging the class of phenomena to which the equilibrium-preserving assumption is applicable. In other words, from the apparently self-evident premises of the equilibrium-preserving assumption a conclusion cannot be derived which is valid for a vaster class of phenomena than that captured by the premises. Vailati's counter-argument, however, has the merit of indirectly showing that, in the case of the equilibrium of the lever, our intuition about the self-evidence of certain premises can be given more precise meaning by means of a functional equation (cf. the Vailati constraint). But the Vailati constraint, when properly investigated, also shows that our intuitions about equilibrium can go wrong.

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[14] The requirement that $F(d)$ increases monotonically translates Archimedes's postulate, according to which the moment of a weight is greater for greater distances from the fulcrum (but note that Archimedes does not use the construct 'moment of a weight' and his wording is different). This postulate is not as intuitive as it might seem at first glance. In a fascinating experimental study of how naïve subjects learn the rules of equilibrium of the balance, Thibodeau et al. (1986) have shown that a bewildering array of heuristic strategies—involving both reasoning and experiment—precedes the grasping of the postulate.

[15] We can assume the existence of the limits since we are interested in investigating physical cases corresponding to locations of weights on the lever, not 'pathological' situations due to non-differentiable functions, which would have been irrelevant for the Mach–Vailati debate.

[16] The whole calculus argument has been suggested to me by John Norton, whom I wish to thank. A more extensive treatment of problems concerning functional equations similar to the Vailati constraint can be found in Aczél (1966), pp. 31 ff.

4. Conclusion: the balance thought experiment

The aim of this final section is to give the reader a perspective by means of a thought experiment that places the earth and its inhabitants in context. Suppose that, instead of the Newtonian inverse square law of gravitation, $F = G\frac{m \cdot M}{r^2}$ (see Appendix), another inverse law obtained, with an arbitrary exponent n (n positive integer), such as $F = G\frac{m \cdot M}{r^n}$. Then, it is easy to verify that the moment of a balance on the surface of imaginary earths, where the field of gravity is in accord with such inverse square laws, would be

$$\mu_n(\alpha) = m \cdot M \cdot G \cdot \frac{R^{1-n} \cdot \sin(\alpha)}{\left[1 + tg(\alpha)^2\right]^{\frac{n}{2}}}$$

instead of the following expression, $\mu(\alpha) = \frac{m \cdot M \cdot G}{R} \cdot \sin(\alpha) \cdot \cos(\alpha)^2$, which is only valid for $n = 2$ (see Appendix).

Let us assume, for instance, n = 2, 4, 8, 16, 32, 64, 128, and call the earths on which such gravity fields obtain E2, E4, E8, E16, E32, E64, E128, respectively (Fig. 15). On these earths, the moments, and therefore the responses of the balances to such transformations as rotations and mass splitting, would all be different. Inhabitants of E128, for example, would have a different experience of the balance—and more generally of equilibrium phenomena—than, say, those of E64, or E2 (ourselves). Let us now ask a Machian question. *What properties of equilibrium could the inhabitants of those earths make up*? They would still allow certain suppositions, such as, for example, that equal weights are in equilibrium at equal distances from the fulcrum of a balance, since the form of universal gravitation they experience still has the central symmetry of the Newtonian form. Indeed, there is a range of parameters (smaller and smaller as n increases) where equilibrium phenomena should look the same on all earths (cf. Fig. 15, where for small angles all curves overlap). However, it is very hard to imagine that the inhabitants of those earths would allow properties of equilibrium to be made up with the help of the equilibrium-preserving assumption, since, as n increases, their balances would tip to one side more and more markedly in response to rotations and mass splitting.

The inhabitants of E2, however, have been granted such a huge range of parameters in which experience does not flout the equilibrium-preserving assumption that for a while they mistook it for a self-evident supposition. It took them quite a journey to figure out that their intuitions were wrong.

Finally, a deep question about the nature of my thought experiment can be noted.[17] Is the nature of my thought experiment empirical or logical (that is, based on experi-
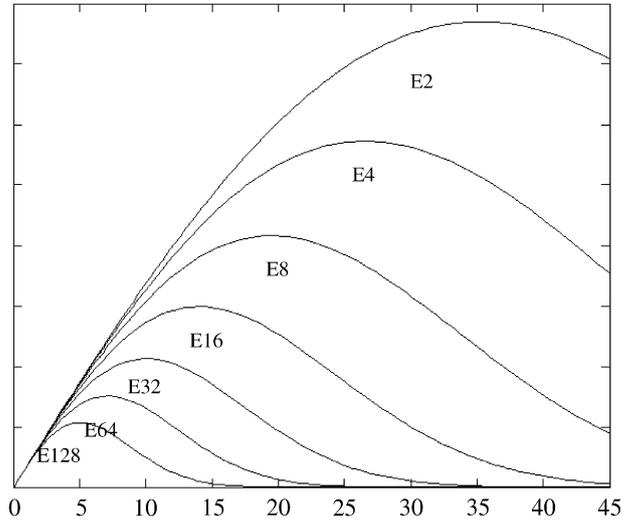

Fig. 15. The moment of a balance on imaginary earths E2, E4, E8, E16, E32, E64, E128. On the horizontal axis the angle α is represented varying in degrees from 0° to 45°. On the vertical axis the value of function $\mu_n(\alpha)$ is represented (scale is irrelevant). For all earths, I have assumed all constants equal to 1.

ence, or based on formal means of reasoning, that is, logical means of reasoning, where I take 'logical' in the Machian sense of self-evident)? In my view, the nature of the thought experiment I proposed is empirical. To be sure, I used equations and calculations, that is, rather refined formal means of reasoning. However, the conclusions that I have reached stand on premises that, up until now, have only been justifiable empirically. For, the mathematical form of universal gravitation that we know of has, still today, no other justification than experience itself (the universe we live in being what it is, or, more precisely, the inverse n law we measure being what it is, that is, n = 2 being acceptable for most practical purposes). In sum, much as I admire the sophisticated formal means of reasoning employed in thought experiments, I feel that the conclusions we can reach are justifiable only insofar as experience allows us to justify the imaginary set-ups in which the thought experiments are carried out.

Appendix

To calculate the condition of equilibrium of a balance in a non-constant gravity field, such as that generated by the earth considered as a perfect sphere, which, as is well known, is equal to the field generated by a punctiform mass placed at the centre of the earth, all we need to do is calculate the moment of a mass m around the fulcrum of the balance (Fig. A1).

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[17] An anonymous referee raised this troubling question. I thank the referee for making me think harder than I had done before about the nature of my thought experiment, even though I am not sure I have attained peace of mind (for, I notice, I am a comfortable empiricist on the majority of the week's days, but pangs of conscience assail me on the other days, when I perform thought experiments). There is a sizeable literature about the logical vs. empirical nature of thought experimentation. It is too well known to the readers of this journal to need mention. Cf., for starters, the debate between John Norton (1996) and James Brown (1991).
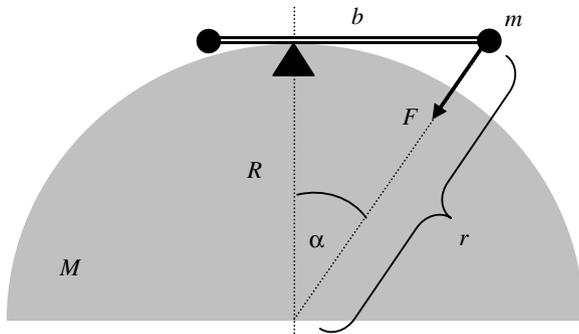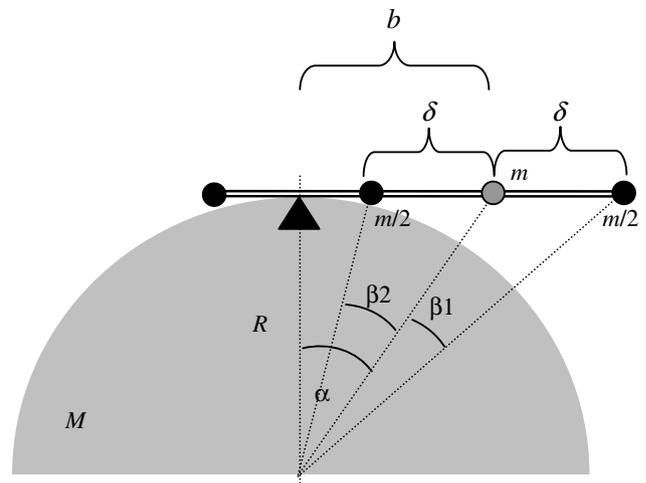
Fig. A1. The condition of equilibrium of a balance in a non-constant gravity field generated by a punctiform mass placed at the centre of the earth. The calculation of the moment of mass m.

Let M be the mass of the earth, R the radius of the earth, r the distance of mass m from the centre of the earth, G the constant of universal gravitation, F the attraction exerted on mass m by the whole earth. Then, we will have the following relations.

$$F = G \frac{m \cdot M}{r^2}$$

$$r = R\sqrt{1 + tg(\alpha)^2}, \quad b = R \cdot tg(\alpha)$$

(by simple trigonometry, cf. Fig.A1).

The moment of a mass m around the fulcrum of the balance is as follows, by definition.

$$\mu(\alpha) = b \cdot F \cdot \cos(\alpha)$$

By replacing in the equation the quantities b and F with their expressions already found, we will have:

$\mu(\alpha) = \frac{m \cdot M \cdot G}{R} \cdot \sin(\alpha) \cdot \cos(\alpha)^2$, which is relation (1) given above.

The moments of the split masses immediately follow from (1), by considering the geometry of how the splitting of the two masses is done (Fig. A2).

By simple trigonometry we can calculate the angles β1, β2, corresponding to the displacement of the split masses at a distance δ from the initial position of mass m. The two relations are as follows.

$$\begin{cases} \frac{b+\delta}{R} = tg(\alpha + \beta1) \\ \frac{b-\delta}{R} = tg(\alpha - \beta2) \end{cases}$$

Then the two moments of the split masses (each of which masses, let us remember, is equal to 1/2 m) will be:

$$\mu(\alpha + \beta1) = \frac{1}{2} \frac{m \cdot M \cdot G}{R} \cdot \sin(\alpha + \beta1) \cdot \cos(\alpha + \beta1)^2$$

$$\mu(\alpha - \beta2) = \frac{1}{2} \frac{m \cdot M \cdot G}{R} \cdot \sin(\alpha - \beta2) \cdot \cos(\alpha - \beta2)^2$$

so that the ratio of the total moment of the split masses to the moment of mass m originally placed at distance b from the fulcrum will be:

Fig. A2. The condition of equilibrium of a balance in a non-constant gravity field generated by a punctiform mass placed at the centre of the earth. The calculation of the moments of the split masses.

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$$= \frac{\sin(\alpha + \beta1) \cdot \cos(\alpha + \beta1)^2 + \sin(\alpha - \beta2) \cdot \cos(\alpha - \beta2)^2}{2\sin(\alpha)\cos(\alpha)^2} .$$

This is the expression used above to draw the diagram (Fig. 13) of the ratio of the moment of two split masses to the moment of a single mass, when the two masses are relocated at equal distances, starting from zero, for an angle $\alpha = 20°$.[18]

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[18] The numerical calculations presented in this paper have been performed with *Mathcad 13,* © 1986–2005 Mathsoft Engineering & Education, Inc.

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