



Discussion

Response to Maarten Van Dyck's commentary

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ABSTRACT

In response to Maarten Van Dyck's commentary, I present a translation of Vailati's original paper with a short introductory note.

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In his intelligent commentary on my paper, Maarten Van Dyck raises important questions to do with the *historical* dimension of the Mach–Vailati debate, which I had set aside in my paper perhaps in rather too cavalier a way. For this timely reminder I am indebted to Maarten Van Dyck. However, Van Dyck's commentary does not seem to be much concerned with the *philosophical-technical* arguments I put forward in my paper, thus failing, in my view, to issue a real challenge to the main thrust of my analysis, to the thought experiment concluding my paper, and ultimately to Mach's claim *that we cannot make up properties of nature with the help of self-evident suppositions, but that these suppositions must be taken from experience*. Rather, it points to a way of reconsidering the Mach–Vailati debate that is, I believe, historically tenable (in that it takes its cue precisely from Giovanni Vailati), and with which I tend to agree *historically*. I will therefore here simply present a translation of the little known paper by Vailati that Van Dyck and I have discussed. This translation—I hope—will be of interest to the readers of this journal, and further stimulate research on this debate.

The Italian historian and philosopher of science, Giovanni Vailati (1863–1909), presented his brilliant paper at the *International Congress of Historical Sciences*, held in Rome in 1903. The paper was published a year later in the *Proceedings* of the conference, and eventually again in the posthumous collection of Vailati's writings edited by M. Calderoni, U. Ricci, and G. Vacca in 1911. Since the latter is, to date, the most extensive and authoritative collection of Vailati's writings, I have based my translation on that edition (see Vailati, 1911, pp. 497–502). Further references related

to Vailati can be found in my previous paper in this journal, Palmieri (2008b), and there is a fine biography of Vailati by Orazio Premoli in Vailati (1911), pp. i–xxix.

The proof of the principle of the lever put forward by Archimedes in the first book of Equilibrium of plane figures

Giovanni Vailati

The recent controversy between Professor Mach and Professor Hölder, concerning the character and the value of the proof of the principle of the lever put forward by Archimedes in his work on the equilibrium of plane figures, seems to offer the opportunity to illustrate, with a concrete example, the usefulness that specialized research on the history of mechanics presents for those who set themselves the task of studying and analysing the methods of research and proof adopted in the sciences mostly based on deductive reasoning.

As is well known, the path followed by Archimedes to arrive at the conclusion that two weights hanging from a beam, free to rotate around a fulcrum, are in equilibrium when they are inversely proportional to their distances from the fulcrum, consists in appealing to the proposition that it is always possible—without altering equilibrium if there was equilibrium previously—to replace a weight applied to any point of the said beam with a couple of weights hanging from two points at equal distances from the point at which the original weight was applied, each weight being equal to one half the weight to be replaced.

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The validity of this proof method has been questioned by Mach in the critical exposition that he gives in his *History of mechanics*.¹ The objection raised by Mach to Archimedes' proof consists in claiming that the above-mentioned premise, on which the proof is based, cannot be deduced from the postulates assumed by Archimedes, and which are enunciated at the beginning of his treatise.

But Mach's objection is not limited to this: Mach thinks that, to accept the above-mentioned premise—in other words, the assumption that shifting the point of application of one half of a given weight does not alter equilibrium when this operation is accompanied by the shift in the opposite direction of the other half of the weight—is tantamount to presupposing that equilibrium can only be altered by displacements that will alter the sum of the products of each weight by the relative distance of its point of application from the fulcrum.

One can counter such a criticism, which tends to qualify Archimedes' proof as circular, by saying that, first of all, the equivalence seen by Mach between the said premise and the principle of the lever is only partial. For, it is true that the displacements of weights that, according to the said premise, are assumed not to alter equilibrium, are among those that the principle of the lever, too, asserts not to alter equilibrium. But the converse is not true, since the principle of the lever considers a larger class of displacements of weights, of which those considered by the premise assumed by Archimedes are only a particular case. From this it follows that the premise of Archimedes' reasoning asserts only a part of the general conclusion at which he arrives; for, by starting from the premise that equilibrium is not altered by a certain class of displacements, he demonstrates that equilibrium is not altered by a larger class of displacements that includes the former class.

In other words, Archimedes' assumption, relating to the condition that must be satisfied for two weights p' , p'' —hanging on the same side with respect to the fulcrum at distances d' , d'' from it—to be in equilibrium with a weight p hanging from a distance d from the fulcrum on the other side of it, can be expressed by an equation of the following form:

$$P' \cdot Fd' + p'' \cdot Fd'' = p \cdot Fd,$$

where F is a function such that, for any length h , the following equation holds:

$$2F \cdot d = F(d + h) + F(d - h).$$

From this premise Archimedes deduces the conclusion that the sought condition is precisely the following:

$$p' \cdot d' + p'' \cdot d'' = p \cdot d.$$

However, to this defence of Archimedes' reasoning against the charge of circularity one can still reply that, even though Archimedes' premises do not presuppose all that is asserted in the conclusion, still something is presupposed that can only be proven by means of the conclusion itself, and which one has no right to consider as being more evident than the conclusion itself.

To this form of Mach's objection, Professor Hölder² answers that Archimedes' premise can be proven by having recourse both to the axioms enunciated by Archimedes, and to another axiom tacitly applied by Archimedes, and which could be formulated as follows: a rigid

body is in equilibrium when the system of forces that act on the body can be regarded as being composed of two parts, each of which is in equilibrium.

But the fact that the said premise, on which Archimedes' proof is based, can be deduced from this axiom, does not allow us to consider the historical question as resolved: that is, whether the premise was obtained by Archimedes in this way or rather by other means.

Now, I believe that this is in fact the case, and that the path that Archimedes followed to arrive at the proposition—that equilibrium is not altered by replacing a weight with two weights, each being equal to one half the weight to be replaced and hanging from equal distances from the point at which the total weight was hanging—is to be found in certain considerations concerning centre of gravity, which Archimedes repeatedly hints at in his demonstrations even though he does not insist on them, as if the question had already been dealt with by him in another work that has not come down to us.

These points, to which I think both Mach and Hölder³ failed to pay attention, refer to the following properties of a rigid body's centre of gravity—the centre of gravity that Archimedes⁴ defines as the point such that, when the body is hanging from it, it will remain in equilibrium regardless of the position in which it is placed:

1. If one imagines a rigid body divided into two parts, the straight line joining the centres of gravity of the parts contains the centre of gravity of the whole body.
2. The position of this centre of gravity on this line depends only on the weight of the parts into which the body has been divided and does not change regardless of how these parts are deformed, as long as the deformation occurs in such a way that their respective centres of gravity do not change position.

The first of these two propositions is explicitly enunciated by Archimedes (Heiberg Edition, II, p. 149), who qualifies it as a theorem that has already been proved.

As for the second, it can be deduced by simply assuming that for any body there exists one and only one centre of gravity, in the sense given above, and by assuming that the only positions in which the body can remain in equilibrium when suspended from a point other than its centre of gravity are those in which the centre of gravity is on the vertical line passing through the point from which the body is suspended.⁵

Let us imagine the body being suspended from its centre of gravity and placed in such a way that the line joining the centres of gravity of the two parts is horizontal. Now, imagine that the two parts no longer have any constraint except for having their centres of gravity connected by a rigid beam. In other words, let us imagine that the system is reduced to a balance of unequal arms from the two ends from which two weights will hang, represented by the two parts into which the body has been divided.

Such a balance will be in equilibrium; for, if it were not, one could determine on the beam another point, different than the original point of suspension, and such that, if the beam were suspended from it, it would remain in equilibrium. If we now imagine that the two parts of the body were once again rigidly connected to each other, there would still be equilibrium. But this would be

¹ Mach (1883). Vailati actually quotes the fourth edition of Mach's book, which was published in 1903.

² Hölder (1900), p. 64.

³ Hölder speaks as strongly as Mach does of the 'Scheinbare Gebrauch' that Archimedes would make of centre of gravity.

⁴ [Translator's note: here Vailati makes a historical error: a definition of centre of gravity is nowhere to be found in Archimedes. In fact, the definition attributed to Archimedes by Vailati is that basically given by Pappus, the only definition of centre of gravity that has come down to us from antiquity. See Palmieri (2008a), p. 314.]

⁵ We can find hints in Book VIII of Pappus' *Collections* pointing to the considerations that may have guided Archimedes towards these two fundamental assumptions. I have already had the opportunity to call attention to this in a note published in *Atti della R. Accademia delle Scienze di Torino* (1897). [Translator's note: this writing was reprinted in Vailati (1911), pp. 79–90.]

incompatible with the fact that the vertical line passing through such a point does not pass through the centre of gravity of the whole body.

Having proved in this way that such a balance of unequal arms remains in equilibrium when it is suspended from the centre of gravity of the whole body, it follows that it will remain in equilibrium even if the two portions of the body are deformed in any way whatever, as long as the centres of gravity of the portions continue to coincide with the ends of the balance. From this it also follows that the deformation of the parts cannot cause a change in the position of the whole body's centre of gravity along the line joining the centres of gravity of the two parts into which the body has been divided. This is what was to be proven.

The simplest way to deduce the principle of the lever from the proposition we have just proven is by applying it to a rectangle that is imagined being divided into two unequal parts by means of a straight line parallel to one of the sides. This is precisely the path followed by Archimedes in his work *On the equilibrium of plane figures*. The core of Archimedes' proof can be found in a most simple construction by Luca Valerio.⁶

On the assumption that, for symmetry reasons, the centre of gravity of a rectangle is on the middle point of one of its diagonals, let us consider rectangle ABCD being divided into two unequal rectangles, ABFE, EFCD. Let the three diagonals be drawn, AC, AE, FC, and let the parallel line to AD be drawn through the middle point of AB. This parallel line will meet the three diagonals at three points L, M, N, the centres of gravity of the three rectangles. Since LN is equal to one half AF and ML to one half EC, it follows that centre of gravity L of the whole rectangle will divide segment MN into parts inversely proportional to the surfaces of the two rectangles into which the rectangle was divided.

If we now imagine the figure being suspended from point L, it will remain in equilibrium and, thanks to what has been proven above, the equilibrium will continue if we imagine the two rectangles being severed from each other and free to rotate around their centres of gravity, as long as these are connected with a rigid beam turning around L. This affords us a lever from whose ends weights are suspended inversely proportional to the distances of their points of suspension from the fulcrum.

These considerations seem to me to suffice to show not only that Archimedes' proof does not deserve the qualification of *petitio principii*, but also why his proof may have appeared as such to an acute and diligent investigator such as Mach. The fact that in our present-day treatment of rational mechanics one arrives at the propositions concerning centres of gravity on which Archimedes' proof is based by following a path along which one makes use of the principle of the lever—or of another more general proposition concerning the composition of forces of which the principle of the lever is only a particular case—naturally predisposes the modern reader to think that this, or a similar one, was the path that was also followed by Archimedes.

Now, this presupposition is far from being confirmed by Archimedes' writings, from which one can see that, on the contrary, the properties of centre of gravity to which he had recourse were for him only consequences of a certain number of fundamental assumptions to which he attributed the same character of intuitive evidence, and the same ability to serve as the basis of proofs, that we attribute to the postulates of geometry or arithmetic.

Some of these mechanical axioms are enunciated by Archimedes at the beginning of the work in which he makes use of them. Others, which are applied less frequently, are hinted at only when they are needed. Others are tacitly used without explicit enunciation, precisely as we see in the geometrical books of Euclid.

A characteristic difference that, aside from their having a greater degree of intuitive evidence, distinguishes these axioms from the propositions derived from them, consists in this: namely, that—as has been noted by Hölder—these axioms refer to most common experiential facts that can be perceived and demonstrated independently of any systematic comparison among the facts that realize them, and without any need for precise and rigorous measurements of the facts, which would otherwise be required if it were necessary to obtain the principle of the lever inductively.

Regardless of one's opinion as to the validity of a treatment of the statics of rigid bodies that starts from Archimedes' fundamental assumptions—or as to the advantages that there can be when the latter are deduced from other more general or more elementary suppositions: for instance, from the principle of virtual work, of which Archimedes' fundamental assumptions would only be a particular set of consequences—it seems to me that such an opinion would be compatible with the recognition of the legitimacy of the method by which Archimedes deduced his conclusions from his premises.

The charge of *Scheinbeweis* levelled at this method—because in it one starts from considering a most particular case of equilibrium, namely that of two equal weights suspended from the two ends of a balance, in order to arrive at the discovery of the conditions of equilibrium for a more general class of cases—does not seem to me more justifiable than, in geometry, to object to a proposition concerning polygons its being based on the properties of triangles.

Finally, the assertion that in Archimedes' premises there is already implicitly contained the conclusion that he deduces from them, seems to me true only in the sense in which it is true of any mathematical proof, insofar as in any mathematical proof the truth of the proposition being proved appears as a simple consequence of certain operations of selection, linking, and coordination performed on fundamental propositions that are the basis of the treatment. And this, to use a celebrated Aristotelian paragon, is as far from diminishing the value of their discovery and of their proof as it would be far from diminishing the value of a sculptor to claim that the statue he created was already within the marble from which he crafted it, and that he has done nothing but to remove the superfluous parts from the block of marble.

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⁶ Valerio (1604), p. 29.