MATH 121 MIDTERM, OCTOBER 15 2014

Name:

Please answer each question in the space given. There is an extra page at the back of the exam for your scrap work, but that work will not be graded.

Please write neatly. If I can't read your answer, I can't give you credit.

Good luck!

Problem	Score	Points Possible
1		15
2		6
3		6
4		6
5		27
6		24
7		16

(1) True or False.

3 (a) The vectors $(1,0,0),(0,2,2),(2,4,4)$ in $\mathbb{R}^3$ are linearly independent.

False

$2(1,0,0)+2(0,2,2)-(2,4,4)=0$ gives a linear dependence.

(b) Let V be a vector space and let $\alpha=\{u_1,...,u_n\}$ be a subset of V. Then α is a basis for V if and only if each vector $v\in V$ can be uniquely expressed as a linear combination of vectors of α.

3

True

α is a basis iff α is linearly independent and is a spanning set for V. Every $v \in V$ can be expressed as a linear combination of vectors in α iff α is a spanning set. This expression is unique iff α is linearly independent.

(c) Let V and W be vector spaces. Let $f: V\to W$ be linear. Moreover, let $\{u_1,...,u_k\}$ be a linearly independent subset of $Range(f)$. Choose $S=\{v_1,...,v_k\}$ in V such that $f(v_i)=u_i$ for $i=1,2,...,k$. Then S is linearly independent.

3

True

Suppose S is not linearly independent. Then $\exists$ linear dependence, not all $a_i=0$ $a_1v_1+\dots+a_kv_k=0$. Then $0=f(a_1v_1+\dots+a_kv_k)=a_1u_1+\dots a_ku_k$. By linear independence of $\{u_1,\dots u_k\}$, all $a_i=0$, contradicting earlier assump that not all $a_i=0$. Hence S must be linearly independent.

(d) If $dim(V)=5$ then V cannot be spanned by four vectors.

3

True

$\dim V=5$ means $\exists$ basis of V w/ 5 elements. In particular, $\exists$ 5 linearly independent vectors in V. By a Proposition in Axler, the size of any linearly independent set is $\leq$ the size of any spanning set. So we cannot have 4 vectors which span V.

(e) $\begin{pmatrix}1\\2\end{pmatrix}$ is an eigenvector of $\begin{pmatrix}3&1\\2&4\end{pmatrix}$.

3

True.

$\begin{pmatrix}3&1\\2&4\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}5\\10\end{pmatrix}=5\begin{pmatrix}1\\2\end{pmatrix}$. So $\begin{pmatrix}1\\2\end{pmatrix}$ is an eigenvector with eigenvalue 5.

(2) Which of the following are fields? $\mathbb{Z}/5\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}, \mathbb{R}, \mathbb{C}, \mathbb{Z}, \mathbb{Q}$?

6

$\mathbb{Z}/5\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$. are fields.

$\mathbb{Z}/6\mathbb{Z}$ is not because some elements do not have multiplicative inverses, eg. 2.

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(3) Which of the following subsets of $\mathbb{R}^4$ are subspaces of $\mathbb{R}^4$?

6

a) $\{(0, x, 2x, 4x) | x \in \mathbb{R}\}$
b) $\{(x^4, x^3, x^2, x) | x \in \mathbb{R}\}$
c) $\{(1, x, y, 0) | x \in \mathbb{R}\}$.

a) is a subspace because it is a subset of $\mathbb{R}^4$, contains $(0,0,0,0)$, is closed under addition and is closed under multiplication by a scalar.

b) is not because it is not closed under addition.

c) is not because it does not contain $(0,0,0,0)$.

(4) Define what it means to say that two vector spaces V, W are isomorphic.

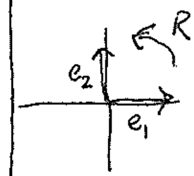
6

Two vector spaces V, W are isomorphic iff $\exists$ an invertible linear transformation $T: V \longrightarrow W$.

(5) Consider the map $R : \mathbb{R}^2 \to \mathbb{R}^2$ given by rotation by 90 degrees counterclockwise.

a) Pick a basis for $\mathbb{R}^2$ and write the matrix representing R in that basis.

(Here's a picture)



8 Pick basis $\{e_1, e_2\}$ of $\mathbb{R}^2$, where $e_1 = (1,0)$, $e_2 = (0,1)$.

Matrix for R with respect to this basis is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

b) Show that R is invertible.

9 ~~A linear map is invertible if ∃ S ∈ L(V,~~

By definition, R is invertible if $\exists\, S \in \mathcal{L}(V,V)$ such that $SR = RS = Id_{\mathbb{R}^2}$.

Let $S = R^3$.

$SR = R^3 R = R^4 = Id$

$RS = RR^3 = R^4 = Id$.

Hence, R is invertible.

Note: this problem can also be done by showing R is injective and surject. In fact, since $R : \mathbb{R}^2 \to \mathbb{R}^2$ it is sufficient to show that R is injective <u>or</u> show that R is surjective.

c) Find a nonzero polynomial satisfied by R which is of minimal degree among all nonzero polynomials satisfied by R. Show the degree of the polynomial you found is, in fact, minimal.

Take $p(x) = x^2 + 1$. R satisfies the polynomial $p(x)$, in other words,

10 $p(R) = 0$. (5)

(There are multiple ways to arrive at $p(x)$. One can compute the matrices for $I, R, R^2, R^3 \ldots$ in terms of the basis $\{e_1, e_2\}$, and find the linear dependence between them by observation. Or, one can observe that $R^4 = I$, so R satisfies the polynomial $q(x) = x^4 - 1$. Since we are looking for a polynomial of minimal degree, factor $q(x)$ (over $\mathbb{R}$!) as $(x^2+1)(x^2-1)$. R satisfies one of the factors, namely x^2+1.)

Now we show that $p(x)$, of degree 2, has minimal degree among all nonzero polynomials satisfied by R.

It suffices to show that R does not satisfy any polynomial of degree 0 or 1. Suppose R did. Then we'd have some polynomial $q(x)$ of degree < 2 s.t.

(5) $q(R) = a_1 R + a_0 I = 0$ for some $a_1, a_0 \in \mathbb{R}$.

If $a_1 = 0$, we have $a_0 I = 0$ which is possible iff $a_0 = 0$.
This would mean $q(x) = 0$, which contradicts the assumption that $q(x)$ is nonzero.

If $a_1 \neq 0$, we have $R = -\frac{a_0}{a_1} I$, ie. that R is a scalar multiple of the identity.

But any $v \in \mathbb{R}^2$ is an eigenvector for a scalar multiple of the identity, while $\binom{0}{1}$ is <u>not</u> an eigenvector for R. So R cannot equal a scalar multiple of the identity.

We conclude that R does not satisfy any polynomial of degree < 2, so the degree of $p(x) = x^2 + 1$ is indeed minimal among the degrees of all nonzero polynomials satisfied by R, as desired.

(6) Let V be a 9 dimensional vector space over a field F.

a) Define the vector space $\mathcal{L}(V,V)$ and give its dimension.

8

6 As a set, $\mathcal{L}(V,V)$ is the set of all linear maps $T: V \to V$.

2 To define addition in $\mathcal{L}(V,V)$, given $T_1, T_2 \in \mathcal{L}(V,V)$, $(T_1+T_2)v = T_1v + T_2v$.
To define multiplication by a scalar, given $T \in \mathcal{L}(V,V)$, $c \in F$,
$(cT)v = c(Tv)$. This defines $\mathcal{L}(V,V)$ as a vector space.

2 $\mathcal{L}(V,V)$ has dimension $(\dim V)^2 = 81$.

b) Define the multiplication law in $\mathcal{L}(V,V)$.

6 For two maps $S, T \in \mathcal{L}(V,V)$ we define $S \cdot T$ as the composition $S \circ T$ of the two maps. That is, $(S \cdot T)v = S(T(v))$.

c) If T in $\mathcal{L}(V,V)$ satisfies $T \cdot T = 0$, show that the dimension of the null space of T is at least 5.

10 We are given a linear map $T \in \mathcal{L}(V,V)$ s.t. $T \cdot T = 0$.
This means $(T \cdot T)v = 0$ for any $v \in V$. By the above definition, $0 = (T \cdot T)v = T(T(v))$, for any $v \in V$.
So null $T \supseteq$ range T. By a Proposition in Axler, if A is finite dimensional and B is a subspace of A, then $\dim B \leq \dim A$.
Hence $\dim \text{null } T \geq \dim \text{range } T$.

On the other hand, by the rank-nullity theorem,
$9 = \dim V = \dim \text{null } T + \dim \text{range } T$.

Hence $9 = \dim \text{null } T + \dim \text{range } T \leq 2 \dim \text{null } T$

Hence $\frac{9}{2} \leq \dim \text{null } T$ or equivalently (since $\dim \text{null } T$ is an integer)

(7) If W is subspace of V, show that $\mathcal{L}(V, W)$ is a subspace of $\mathcal{L}(V, V)$.

16

To show that $\mathcal{L}(V,W)$ is a subspace of $\mathcal{L}(V,V)$, we must show that

1. $\mathcal{L}(V,W)$ is a subset of $\mathcal{L}(V,V)$
2. $\mathcal{L}(V,W)$ contains 0
3. $\mathcal{L}(V,W)$ is closed under addition
4. $\mathcal{L}(V,W)$ is closed under scalar multiplication.

We prove them here:

(1) Given any $T \in \mathcal{L}(V,W)$, T is a linear map from V to W. Since $W \subseteq V$, T can equally well be considered as a map from

4 V to V (whose image happens to land inside W) and this map is still linear. So T can also be considered as an element of $\mathcal{L}(V,V)$. Hence $\mathcal{L}(V,W)$ can be considered as a subset of $\mathcal{L}(V,V)$.

(2) The map $0: V \to W$ that sends any $v \in V$ to $0 \in W$ is the additive identity in $\mathcal{L}(V,W)$. We rely here on the assumption that W is

2 a subspace of V and hence contains 0 to know that the zero map exists in $\mathcal{L}(V,W)$.

(3) Given $T_1, T_2 \in \mathcal{L}(V,W)$, we defined $(T_1+T_2)v = T_1v + T_2v$.

5 Since W is a subspace, W is closed under addition. Hence $T_1v, T_2v \in W$ $\Rightarrow T_1v + T_2v \in W$. Hence $T_1+T_2 \in \mathcal{L}(V,W)$, as desired.

(4) Given $a \in F$, $T \in \mathcal{L}(V,W)$, $(aT)v = a(Tv)$ by definition. Because W is a subspace of V, W is closed under scalar multiplication.

5 Hence if $Tv \in W$ then $a(Tv) \in W$

For any $v \in V$, $Tv \in W$ because $T \in \mathcal{L}(V,W)$. We conclude for any $v \in V$ $(aT)v \in W$, hence $aT \in \mathcal{L}(V,W)$, as desired.

Page for scratchwork. Nothing on this page will be graded.