MATH 121

Homework 8, due October 29

Note: For this homework, please use just the material we have covered in class. There are multiple ways to solve several of the problems and we will see others soon, but for now the intention is to have you familiarize yourself with the techniques covered thus far.

Part I

(1) Axler, page 96, Problem 23.

(2) Axler, page 96, Problem 24.

(3) Consider the map $R : \mathbb{C}^2 \to \mathbb{C}^2$ given by rotation by 90 degrees counterclockwise. Compute the eigenvalues and eigenvectors of R. Specifically, find eigenvalues and eigenvectors of R, and then show that there can be no more.

Why did I not ask you about the more familiar operator $R : \mathbb{R}^2 \to \mathbb{R}^2$?

(4) Consider the linear operator $T \in \text{Ł}(\mathbb{R}^3)$ whose matrix with respect to the standard basis is $\begin{pmatrix} 2 & -2 & 3 \\ -2 & -1 & 6 \\ 1 & 2 & 0 \end{pmatrix}$.

a) Find a nonzero polynomial satisfied by T of minimal degree among all nonzero polynomials satisfied by T.

b) What are the only possible eigenvalues of T?

c) Can you find a basis of eigenvectors of T?

Part II

(5) Let V be a finite dimensional vector space of dimension n, and consider a linear operator $T \in \text{Ł}(V, V)$. Suppose that T is nilpotent, i.e. that there exists some natural number m such that $T^m = 0$. Show that, in fact, $T^n = 0$.

(6) Let V be a vector space of dimension n over F and let k be an integer with $1 < k < n$. Give an example of a linear operator $T : V \to V$ which satisfies the polynomial $p(x) = x^k$ but does not satisfy the polynomial $p(x) = x^{k-1}$.

(7) Consider the linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ given in the standard basis by the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

a) Find all eigenvalues and eigenvectors of T.
b) Find all eigenvalues and eigenvectors of T^2.
c) Find all eigenvalues and eigenvectors of T^3.