12. Problem: Prove that the vector space F^S of functions from a set S is finite-dimensional if and only if the set S is finite.

Solution. For each $s \in S$, let χ_s be the function

$$\chi_s(t) = \begin{cases} 1, & s = t \\ 0, & s \neq t. \end{cases}$$

Assume first that S is finite. We claim that the finite set $\{\chi_s | s \in S\}$ spans F^S. Given $f \in F^S$, we have

$$f = \sum_{s \in S} f(s)\chi_s$$

since, when applying the right side to an input $t \in S$, the $s = t$ term contributes $f(t)$ and the other terms are 0. So we have a finite spanning set and thus F^S is finite-dimensional.

Now assume that F^S is finite-dimensional, say $\dim F^S = n$, but S is infinite. We will produce a contradiction by finding $n+1$ linearly independent vectors in F^S. Let $s_1, \ldots, s_{n+1}$ be any distinct elements of S, and consider the corresponding functions χ_{s_i}. Suppose that for some $a_1, \ldots, a_{n+1} \in F$, we have the equality of functions

$$\sum_{i=1}^{n+1} a_i \chi_{s_i}(t) = 0.$$

Plug $t = s_j$. Then all terms on the left vanish except the one where $i = j$, yielding $a_j = 0$. Thus $\chi_{s_1}, \ldots, \chi_{s_{n+1}}$ are $n+1$ linearly independent vectors in an n-dimensional vector space, which is a contradiction.