Math 121 Practice Final Solutions

December 9, 2014

Email me at `odorney@college.harvard.edu` with any typos.

1. True or False.

   (a) If B is a 6×6 matrix with characteristic polynomial $\lambda^3(\lambda-3)^2(\lambda+5)$, then rank(B) is at least 3.

   Solution. True. The factors $(\lambda-3)^2(\lambda+5)$ indicate that the direct sum U of the generalized eigenspaces corresponding to 3 and -5 has dimension 3. Since 0 is not an eigenvalue of $B|_U$, we get that B acts invertibly on U, so $U \subseteq \operatorname{range} B$.
   (Or) Write B in Jordan normal form (taking it to be a linear operator on $\mathbb{C}^6$). After permuting the blocks, we have

   $$B = \begin{pmatrix} 3 & & & & & * \\ & 3 & & & & \\ & & -5 & & & \\ & & & 0 & & \\ & & & & 0 & \\ 0 & & & & & 0 \end{pmatrix}$$

   The first three columns lie in the range of B and are clearly linearly independent.

   (b) If D and E are matrices satisfying $DE = ED$ and D is diagonalizable (over the complex numbers), then E must be also diagonalizable (over the complex numbers).

   Solution. False. Take D to be the identity and E to be the nondiagonalizable matrix

   $$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

   (c) Let $T : V \to V$ be a linear operator. If the characteristic polynomial of T is equal to the minimal polynomial of T, then T is diagonalizable.

   Solution. False. The matrix E above has characteristic and minimal polynomials both λ^2, but it is not diagonalizable.

2. Find a 3×3 matrix $D : \mathbb{R}^3 \to \mathbb{R}^3$ whose null space consists of all vectors of the form $\begin{pmatrix} r \\ r \\ r \end{pmatrix}$, and whose range consists of all vectors of the form $\begin{pmatrix} s+t \\ s-t \\ 2t \end{pmatrix}$ (Here r, s, t are arbitrary real numbers.)

   Solution. One possible answer:

$$D = \begin{pmatrix} 1 & 1 & -2 \\ 1 & -1 & 0 \\ 0 & 2 & -2 \end{pmatrix},$$

   an operator on $\mathbb{R}^3$. It is clear that D sends $(s, t, 0)$ to $(s+t, s-t, 2t)$ and (r, r, r) to zero; by Rank-Nullity, neither the range nor the null space is any larger.

3. One of the ways we defined the determinant of a matrix was as the unique function from $Mat_{n \times n}(F) \to F$ that satisfies three properties. List those properties.

   Solution.

   - It is linear in each column;
   - If any two columns are equal, the determinant is zero. (Equivalently, if any two *adjacent* columns are equal, the determinant is zero.)
   - The determinant of the identity matrix is 1.

4. Define the following:

   - Vector space

     Solution. A *vector space over a field F* is a set V together with two operations $+ : V \times V \to V$ and $\cdot : F \times V \to V$ satisfying the following conditions (where $a, b \in F$ and $u, v, w \in V$):

     (1) $u + v = v + u$;
     (2) $u + (v + w) = (u + v) + w$;
     (3) There is an element $0 \in V$ such that $u + 0 = u$ for all $u \in V$;
     (4) For each $u \in V$, there is an element $-u \in V$ such that $u + (-u) = 0$;
     (5) $(ab)u = a(bu)$;
     (6) $1u = u$;
     (7) $a(u + v) = au + av$;
     (8) $(a + b)u = au + bu$.

   - Direct sum

Solution. A vector space V is the *direct sum* of subspaces $U_1, U_2, \dots, U_n$ if every vector in V can be written uniquely as a sum $u_1 + u_2 + \cdots + u_n$ with $u_i \in U_i$.

5. Fix a field F and let V be a vector space over F and $T \in \mathcal{L}(V, V)$ a linear operator. Fix two elements $a, b \in F$. Prove that $(T - aI)$ commutes with $(T - bI)$.

Solution. We have

$$\begin{aligned}(T - aI)(T - bI) &= T \cdot T - T \cdot bI - aI \cdot T - aI \cdot bI \\ &= T^2 - bT - aT + abI \\ &= T^2 - aT - bT + baI \\ &= T \cdot T - T \cdot aI - bI \cdot T - bI \cdot aI \\ &= (T - bI)(T - aI).\end{aligned}$$

6. Suppose $A \in Mat_{7\times 7}(\mathbb{C})$ is a matrix with characteristic polynomial $p(z) = (z-2)^2(z-3)^2(z-4)^3$. Suppose as well that $dim(Null(A - 2I)) = 1$, $dim(Null(A - 3I)) = 2$, and $dim(Null(A - 4I)) = 1$.

Find a matrix which is the Jordan canonical form of A. Is the matrix you found unique?

Solution. The datum $\dim \operatorname{null}(A - \lambda I)$ tells us the number of linearly independent eigenvectors corresponding to λ, which equals the number of Jordan blocks with eigenvalue λ. Also, the multiplicities (the exponents in $p(z)$) show the total number of appearances of each λ on the diagonal. Consequently we must have

- for $\lambda = 2$, one block of size 2;
- for $\lambda = 3$, two blocks of total size 2;
- for $\lambda = 4$, one block of size 3.

So we can construct the Jordan form:

$$\begin{pmatrix} 2 & 1 & & & & & \\ & 2 & & & & & \\ & & 3 & & & & \\ & & & 3 & & & \\ & & & & 4 & 1 & \\ & & & & & 4 & 1 \\ & & & & & & 4 \end{pmatrix}$$

It is unique up to permuting the blocks.

7. Consider $B = \begin{pmatrix} 9 & k \\ k & 1 \end{pmatrix}$, where k is a real number.

   (a) For which values of k is B invertible?

Solution. Determinants to the rescue!

$$\det B = 9 - k^2 = -(k+3)(k-3).$$

Thus B is invertible for all k except ± 3.

(b) For which values of k is B diagonalizable?

Solution. All k; see proof after part (c).

(c) Find the eigenvalues of the matrix B. Your answer should be in terms of k.

Solution. The characteristic polynomial is

$$\det(\lambda I - B) = (\lambda - 1)(\lambda - 9) - k^2 = (\lambda - 5)^2 - 16 - k^2.$$

Thus the eigenvalues are $\lambda = 5 \pm \sqrt{k^2 + 16}$.

Notice that $k^2 + 16 > 0$ for all real k. Therefore, the eigenvalues are always distinct, and B is always diagonalizable.

8. Let T be a linear operator over C with characteristic polynomial $(z - \lambda_1) \cdots (z - \lambda_n)$, and let $f(z)$ be a polynomial. Prove that the characteristic polynomial of $f(T)$ is $(z - f(\lambda_1)) \cdots (z - f(\lambda_n))$.

Solution. Write T in upper triangular form (e.g. Jordan normal form). Then the diagonal entries are the eigenvalues repeated according to multiplicity, and WLOG we may take them to be $\lambda_1, \ldots, \lambda_n$ in order:

$$T = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

By matrix multiplication,

$$T^i = \begin{pmatrix} \lambda_1^i & & * \\ & \ddots & \\ 0 & & \lambda_n^i \end{pmatrix} \quad \text{so} \quad p(T) = \begin{pmatrix} p(\lambda_1) & & * \\ & \ddots & \\ 0 & & p(\lambda_n) \end{pmatrix}$$

simply by taking a linear combination. Consequently, the characteristic polynomial of $p(T)$ is $(z - p(\lambda_1)) \cdots (z - p(\lambda_n))$.

9. Let A be the following matrix:

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

Find the determinant, rank, characteristic polynomial, the eigenvalues and the eigenvectors of A. Find an invertible matrix Q such that $D = Q^{-1}AQ$ is diagonal. Compute D.

Solution. **Note: a typo in the original formulation made the problem impractical to solve.**

The rank is 2, since the first two columns are linearly independent and the third is minus their sum. Since A is not invertible, its determinant is therefore 0.

Next we find the characteristic polynomial:

$$
\begin{aligned}
p(t) = \det \begin{pmatrix} t-1 & 1 & 0 \\ 1 & t-2 & 1 \\ 0 & 1 & t-1 \end{pmatrix} \\
&= (t-1)(t-2)(t-1) - (t-1) - (t-1) \\
&= (t-1)(t^2-3t) \\
&= t(t-1)(t-3).
\end{aligned}
$$

Thus 0, 1, and 3 are eigenvalues, each with multiplicity 1. The corresponding eigenvectors are not hard to discover:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ for } 0, \quad \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \text{ for } 1, \quad \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \text{ for } 3.$$

Consequently, the change-of-basis matrix $Q = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{pmatrix}$ yields a diagonal form $D = Q^{-1}AQ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

10. Consider the linear transformation $T : \mathbb{R}[x] \to \mathbb{R}[x]$ given by $T(f)(x) = f(x) + f(-x)$. Describe the null space of T and the range of T as subspaces of $\mathbb{R}[x]$.

Solution. Padding with a zero term if necessary, we can write any polynomial f as $f(x) = a_0 + a_1x + \cdots + a_{2m+1}x^{2m+1}$. We then compute

$$
\begin{aligned}
T(f)(x) &= (a_0 + a_1x + \cdots + a_{2m}x^{2m} + a_{2m+1}x^{2m+1}) + (a_0 - a_1x + \cdots + a_{2m}x^{2m} - a_{2m+1}x^{2m+1}) \\
&= 2(a_0 + a_2x^2 + \cdots + a_{2m}x^{2m}.
\end{aligned}
$$

We then see that

(a) The null space of T is the set of all polynomials $a_1x + a_3x^3 + \cdots a_{2m+1}x^{2m+1}$ that involve only odd powers of x.

(b) The range of T is the set of all polynomials $a_0 + a_2x^2 + \cdots + a_{2m}x^{2m}$ that involve only even powers of x (as the factor of 2 can be disregarded without affecting the range).

11. The Fibonacci sequence is defined by the recursive formula $a_1 = 1, a_2 = 1$, and $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 3$. Compute a non-recursive formula for a_n as a function of n. *Hint*: use diagonalization to compute A^n for a suitable matrix A.

Solution. Observing that

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix},$$

we set about diagonalizing the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. First note that the characteristic polynomial is $\lambda^2 - \lambda - 1$, so the eigenvalues are

$$\varphi = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \tilde{\varphi} = \frac{1-\sqrt{5}}{2}.$$

Each has multiplicity 1; it is easy to compute that the respective eigenvectors are

$$v_1 = \begin{pmatrix} \varphi \\ 1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} \tilde{\varphi} & 1 \end{pmatrix}.$$

To compute F_n, we may apply A^n to the vector

$$\begin{pmatrix} F_1 \\ F_0 \end{pmatrix} = e_1.$$

Thye computation of $A^n e_1$ using the eigenbasis runs through the following steps:

(1) *Convert e_1 to the eigenbasis.* We have $v_1 - v_2 = (\varphi - \tilde{\varphi})e_1 = \sqrt{5} \cdot e_1$ so

$$e_1 = \frac{v_1 - v_2}{\sqrt{5}}$$

(2) *Apply A^n.* Since v_1 and v_2 are eigenvectors, we get

$$A^n e_1 = \frac{\varphi^n v_1 - \tilde{\varphi}^n v_2}{\sqrt{5}}.$$

(3) *Convert back to the standard basis.* We get

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = A^n e_1 = \frac{\varphi^n}{\sqrt{5}} \begin{pmatrix} \varphi \\ 1 \end{pmatrix} - \frac{\tilde{\varphi}^n}{\sqrt{5}} \begin{pmatrix} \tilde{\varphi} \\ 1 \end{pmatrix},$$

of which we are interested only in the second coordinate

$$F_n = \frac{\varphi^n - \tilde{\varphi}^n}{\sqrt{5}} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

This is the usual explicit formula for Fibonacci numbers.

12. $\mathbb{C}$ can be considered as a vector space over $\mathbb{C}$ and over $\mathbb{R}$.

(a) What is the dimension of $\mathbb{C}$ when considered a vector space over $\mathbb{C}$? Over $\mathbb{R}$?

Solution. $\mathbb{C}$ is one-dimensional over $\mathbb{C}$ and two-dimensional over $\mathbb{R}$.

(b) Prove that $S : \mathbb{C} \to \mathbb{C}$ given by $S(z) = \overline{z}$ is additive but not $\mathbb{C}$-linear. Show that it is $\mathbb{R}$-linear and compute its matrix with respect to the ordered basis $\{1, i\}$.

Solution. If $z = a + bi$ and $w = c + di$, then

$$\begin{aligned}
\overline{z+w} &= \overline{(a+c)+(b+d)i} \\
&= (a+c)-(b+d)i \\
&= (a-bi)+(c-di) \\
&= \bar{(}z) - \bar{(}w).
\end{aligned}$$

Thus S is additive. For S to be $\mathbb{C}$-linear, we would need $\overline{zw} = z \cdot \overline{w}$, which is not true (take $z = i$ and $w = 1$).

However, S is $\mathbb{R}$-linear since, for $c \in \mathbb{R}$ and $z = a + bi \in \mathbb{C}$,

$$\begin{aligned}
\overline{cz} &= \overline{ca + cbi} \\
&= ca - cbi \\
&= c(a - bi) \\
&= c \cdot \overline{z}.
\end{aligned}$$

S sends the basis $\{1, i\}$ to $\{1, -i\}$ and thus has the matrix form

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$