More Continued fractions, Approximations.

October 11, 2012

1 Reading Assignment:

1. Read [**I-R**] Chapter 13, section 1.
2. Suggested reading: Davenport: *All* of Chapter IV.

2 Homework set due October 18

1. **I-R**] Page 295 Exercises 6, 11, 13, 15
2. Let n be a positive integer. Find the continued fraction expansion of
   (a) $\sqrt{n^2+1}$
   (b) $\sqrt{n(n+1)}$. (Hint: $n, 2, \ldots$)
3. Prove that for any real irrational number α there are infinitely many fractions $\frac{x}{y}$ such that $|\alpha - \frac{x}{y}| < \frac{1}{y^2}$.

3 The 'general' continued fraction

We now take the 'terms' a_i to be independent variables, and call them q_i.

We'll study the structure of the rational function of the q_i's:

$$q_0 + \frac{1}{q_1+}\frac{1}{q_2+}\frac{1}{q_3+}\cdots$$

So, writing

$$\frac{P_n}{Q_n} = q_0 + \frac{1}{q_1+}\frac{1}{q_2+}\cdots\frac{1}{q_n}$$

Here P_n and Q_n are in $\mathbf{Z}[q_0, q_1, \ldots, q_n]$ and the fraction $\frac{P_n}{Q_n}$ is in 'lowest terms.' We have:

$$P_0 = q_0;\ Q_0 = 1;\ P_1 = q_0 q_1 + 1;\ Q_1 = q_1,$$

and going forward:

$$q_0 + \frac{1}{q_1 +} \frac{1}{q_2} = q_0 + \frac{q_2}{q_1 q_2 + 1} = \frac{q_0 q_1 q_2 + q_0 + q_2}{q_1 q_2 + 1}.$$

SO,

$$P_2 = q_0 q_1 q_2 + q_0 + q_2; \quad Q_2 = q_1 q_2 + 1.$$

Go once more:

$$P_3 = q_0 q_1 q_2 q_3 + q_0 q_1 + q_0 q_3 + q_2 q_3 + 1.$$

$$Q_3 = q_1 q_2 q_3 + q_1 + q_3.$$

3.1 New notation!

Definition 1

$$P_n = [q_0, q_1, q_2, \ldots, q_n] \quad \in \quad Z[q_0, q_1, q_2, \ldots, q_n].$$

3.2 Recurence relations

Proposition 1

$$Q_n = [q_1, q_2, q_3, \ldots, q_n] \quad \in \quad Z[q_0, q_2, q_3, \ldots, q_n].$$

Proposition 2

$$(*) \qquad [q_0, q_1, q_2, \ldots, q_n] = q_0[q_1, q_2, q_3, \ldots, q_n] + [q_2, q_3, \ldots, q_n].$$

Proof:

$$\frac{[q_0, q_1, q_2, \ldots, q_n]}{[q_1, q_2, q_3, \ldots, q_n]} = q_0 + 1/\frac{[q_1, q_2, \ldots, q_n]}{[q_2, q_3, \ldots, q_n]}$$

So, multiply by $[q_1, q_2, q_3, \ldots, q_n]$ to get

$$[q_0, q_1, q_2, \ldots, q_n] = q_0[q_1, q_2, \ldots, q_n] + [q_2, q_3, \ldots, q_n].$$

Remarks.

1. Interpret the empty square bracket [ ] as 1 to get the recurrence relation valid beginning with $n = 2$.
2. The *polynomials* $[q_0, q_1, q_2, \ldots, q_n]$ and $[q_1, q_2, q_3, \ldots, q_n]$ have no common factors, but more interesting to us is that when one substitutes integer values $a_i > 0$ for the q_i's the *integers* $[a_0, a_1, a_2, \ldots, a_n]$ and $[a_1, a_2, a_3, \ldots, a_n]$ are also relatively prime. This will be proved (Theorem 4 below).

3.3 Euler's Rule

$[q_0, q_1, q_2, \ldots, q_n]$ *is the polynomial you get as a sum of monomials obtained by casting out collections of consecutive pairs of variables from the product*

$$q_0 q_1 q_2 \cdots q_n.$$

So,

$$[q_0, q_1, q_2, q_3] = q_0q_1q_2q_3 + q_0q_1 + q_0q_3 + q_2q_3 + 1,$$

and

$$[q_0, q_1, q_2, q_3, q_4] = q_0q_1q_2q_3q_4 + q_0q_1q_2 + q_0q_1q_4 + q_0q_3q_4 + q_2q_3q_4 + q_0 + q_2 + q_4.$$

Euler's Rule is proved directly from the recurrence relation!

Corollary 1 ('reversal')

$$[q_0, q_1, q_2, q_3, \ldots, q_n] = [q_n, q_{n-1}, q_{n-2}, q_{n-3}, \ldots, q_1, q_0].$$

Corollary 2

$$[q_0, q_1, q_2, q_3, \ldots, q_n] = q_n[q_{n-1}, q_{n-2}, \ldots, q_0] + [q_{n-2}, q_{n-3}, \ldots, q_0] =$$

$$= q_n[q_0, q_1, \ldots, q_{n-1}] + [q_0, q_1, \ldots, q_{n-2}].$$

That is, returning to the old notation:

$$P_n := [q_0, q_1, q_2, q_3, \ldots, q_n],$$

and

$$Q_n := [q_1, q_2, q_3, \ldots, q_n],$$

so that the n-th convergent is $\frac{P_n}{Q_n}$ we have these recurrence relations for the numerators and denominators:

Corollary 3

$$P_n = q_n P_{n-1} + P_{n-2},$$

and

$$Q_n = q_n Q_{n-1} + Q_{n-2},$$

Comment on Fibonnaci and generalized Fibonacci.

Theorem 4 (The "gcd equals 1" Theorem)

$$P_n Q_{n-1} - Q_n P_{n-1} = (-1)^{n+1}.$$

Proof: Induction! Check for $n = 2$, and then:

$$P_n Q_{n-1} = q_n P_{n-1} Q_{n-1} + P_{n-2} Q_{n-1},$$

while

$$Q_n P_{n-1} = q_n P_{n-1} Q_{n-1} + Q_{n-2} P_{n-1}.$$

4 Back to numerical continued fractions

A consequence of the "gcd equals 1" Theorem (Theorem 4 of the previous section) is that the n-th convergents are displayed, via the above theory, as fractions in lowest terms, in the sense that the numerators and the denominators of the n-th convergents, as given by the P's and Q's are relatively prime, even when one substitutes natural numbers for the q_i's.

Another immediate consequence (dividing by $Q_{n-1}Q_n$) is:

Corollary 5

$$\frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}} = \frac{(-1)^{n+1}}{Q_{n-1}Q_n}.$$

In this section, we assume that we have substituted natural numbers a_i for the q_i but still denote by $P_n =\in \mathbf{Z}$ and $Q_n \in Z^{\geq 1}$ the values:

$$P_n := [a_0, a_1, a_2, a_3, \dots, a_n] \in \mathbf{Z},$$

and

$$Q_n := [a_1, a_2, a_3, \dots, a_n] \in \mathbf{Z}.$$

Now let $\alpha \in \mathbf{R}$ be an irrational number whose *terms* in its continued fraction expansion given by the natural numbers

$$a_1, a_2, a_3, \dots, a_n, \dots$$

Discuss

- the growth of the numbers P_n, Q_n, and
- alternation of sign of the differences:

$$\alpha - \frac{P_n}{Q_n}.$$

4.1 An application of 'reversal'

Again let $\alpha \in \mathbf{R}$ be an irrational number whose *terms* in its continued fraction expansion given by the natural numbers

$$a_1, a_2, a_3, \dots, a_n, \dots$$

We have:

$$\alpha = \frac{[a_0, a_1, a_2, a_3, \ldots, a_n, \alpha_{n+1}]}{[a_1, a_2, a_3, \ldots, a_n, \alpha_{n+1}]}.$$

Recall Corollary 1:

$$[q_0, q_1, q_2, q_3, \ldots, q_n, q_{n+1}] = q_{n+1}[q_0, q_1, \ldots, q_n] + [q_0, q_1, \ldots, q_{n-1}].$$

So,

$$[a_0, a_1, a_2, a_3, \ldots, a_n, \alpha_{n+1}] = \alpha_{n+1}[a_0, a_1, \ldots, a_n] + [a_0, a_1, \ldots, a_{n-1}] =$$

$$= \alpha_{n+1}P_n + P_{n-1}.$$

and, of course:

$$[a_1, a_2, a_3, \ldots, a_n, \alpha_{n+1}] = \alpha_{n+1}[a_1, \ldots, a_n] + [a_1, \ldots, a_{n-1}] =$$
$$= \alpha_{n+1}Q_n + Q_{n-1}.$$

Therefore:

$$\alpha = \frac{[a_0, a_1, a_2, a_3, \ldots, a_n, \alpha_{n+1}]}{[a_1, a_2, a_3, \ldots, a_n, \alpha_{n+1}]} =$$

$$= \frac{\alpha_{n+1}P_n + P_{n-1}}{\alpha_{n+1}Q_n + Q_{n-1}}.$$

To record this:

Theorem 6

$$\alpha = \frac{\alpha_{n+1}P_n + P_{n-1}}{\alpha_{n+1}Q_n + Q_{n-1}}.$$

Corollary 7

$$|\alpha - \frac{P_n}{Q_n}| < \frac{1}{Q_nQ_{n+1}} < \frac{1}{Q_n^2}.$$

Proof: Compute:

$$\alpha - \frac{P_n}{Q_n} = \frac{\alpha_{n+1}P_n + P_{n-1}}{\alpha_{n+1}Q_n + Q_{n-1}} - \frac{P_n}{Q_n} =$$

$$= \frac{P_{n-1}Q_n - Q_{n-1}P_n}{Q_n(\alpha_{n+1}Q_n + Q_{n-1})} \;=\; \frac{\pm 1}{Q_n(\alpha_{n+1}Q_n + Q_{n-1})}.$$

But since $\alpha_{n+1} > a_{n+1}$ and $Q_{n+1} = a_{n+1}Q_n + Q_{n-1}$, we get:

$$|\alpha - \frac{P_n}{Q_n}| < \frac{1}{Q_n Q_{n+1}} < \frac{1}{Q_n^2}.$$