Introduction to Dirichlet series, the Zeta-function, Dirichlet product

September 11, 2012

1 Recall from last time:

Theorem 1 *The sum of the reciprocals of prime numbers,*

$$\sum_{p \text{ prime}} 1/p,$$

diverges.

Proof on page 21 section 3 of Chapter 2 of **[I-R]**. The idea is to form the finite product

$$\lambda(N) := \prod_{p \text{ prime } \leq N} \frac{1}{1-\frac{1}{p}}.$$

That is,

Lemma 1 *Letting* $p_1, p_2, \ldots p_{\pi(N)}$ *be the set of primes* $\leq N$ *we have:*

$$\lambda(N) = \sum (p_1^{a_1} p_2^{a_2} p_3^{a_3} \ldots p_{\pi(N)}^{a_{\pi(N)}})^{-1}$$

where the sum is over all tuples of non-negative integers $(a_1, a_2, \ldots, a_{\pi(N)}))$.

Since any number $n \leq N$ is expressible as such a $p_1^{a_1} p_2^{a_2} p_3^{a_3} \ldots p_{\pi(N)}^{a_{\pi(N)}}$ we have that

$$\sum_{n \leq N} 1/n \leq \lambda(N)$$

which gives, for example, that

$$\lambda(N) \to \infty$$

(and therefore we already have as consequence that there are infinitely many primes).

Now form

$$\log(\lambda(N)) = -\sum_{p \text{ prime } \leq N} \log(1 - \frac{1}{p})$$

and break up each summand above as:

$$-\log(1 - \frac{1}{p}) \ = \frac{1}{p} + \epsilon_p,$$

where

$$\epsilon_p := \{\frac{1}{2p^2} + \frac{1}{3p^3} + \frac{1}{4p^4} + \dots\} \quad \leq \quad \{\frac{1}{p^2} + \frac{1}{p^3} + \frac{1}{p^4} + \dots\} = p^{-2}/(1 - p^{-1}) \ \leq 2/p^2.$$

So,

$$\log(\lambda(N)) = \sum_{p \text{ prime } \leq N} \frac{1}{p} + \sum_{p \text{ prime } \leq N} \epsilon_p.$$

Now, $\sum_{p \text{ prime}} \epsilon_p$ converges, say, to $\epsilon < \infty$ and we have

$$\sum_{p \text{ prime } \leq N} 1/p \ \geq \log(\lambda(N)) - \epsilon.$$

Discuss this "$\log\log(N)$" type divergence.

1.1 Infinite Product Expansion of $\zeta(s)$

One way of expressing the **Unique Factorization Theorem** for **Z** is that we have the following equality between the formal sum on the RHS and the formal product on the LHS:

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^s}}.$$

Discuss.

1.2 Arithmetic Functions related to $\zeta(s)$

By an *arithmetic function* let us just mean a function $n \mapsto f(n)$ on positive integers with values in **Z**, (or **R** or **C**, but below our functions will have values in non-negative integers).

Definition 1 *A* **multiplicative** *arithmetic function is an arithmetic function $n \mapsto f(n)$ such that*

$$f(a \cdot b) = f(a) \cdot f(b)$$

if a and b are relatively prime.

In contrast, one might call a **completely multiplicative arithmetic function** one such that $f(a \cdot b) = f(a) \cdot f(b)$ for arbitrary pairs a, b.

For any integer m let $\mathrm{Div}(m)$ denote the set of (positive) divisor of m. Here is a useful observation:

If $m = a \cdot b$ with a and b relatively prime, any divisor of m factors uniquely as a product of a divisor of a and a divisor of b. In symbols, if $d \mid m$, then d can be written uniquely as

$$d = d_a \cdot d_b,$$

where $d_a \mid a$ and $d_b \mid b$. Moreover the rule

$$d \mapsto (d_a, d_b)$$

gives a one:one correspondence between $Div(a \cdot b)$ and the cartesian product $Div(a) \times Div(b)$ (provided, of course, that a and b are relatively prime).

1. The divisor function:

$$\nu(n) := \text{the number of positive divisors of } n \quad = \quad \sum_{1 \le d \mid n} 1.$$

Proposition 1 *(a) If* $n = \prod_{p \text{ prime}} p^e$ *then*

$$\nu(n) = \prod_{p \text{ prime}} (e+1).$$

(b) The arithmetic function $n \mapsto \nu(n)$ *is multiplicative.*

(c)

$$\zeta(s)^2 = \sum_{n=1}^{\infty} \frac{\nu(n)}{n^s}.$$

2. The sum-of-divisors function:

$$\sigma(n) := \text{the sum of positive divisors of } n \quad = \quad \sum_{1 \le d \mid n} d.$$

Proposition 2 *(a) If* $n = \prod_{p \text{ prime}} p^e$ *then*

$$\sigma(n) = \prod_{p \text{ prime}} (p^{e+1} - 1).$$

(b) The arithmetic function $n \mapsto \sigma(n)$ *is multiplicative.*

(c)

$$\zeta(s)\zeta(s-1) = \sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s}.$$

3. The sum-of-k-th powers-of-divisors function:

$$\sigma_k(n) :== \sum_{1 \le d \ | \ n} d^k.$$

Small exercise: Frame and prove a proposition for σ_k that is analogous to Propositions 1 and 2. In particular, prove:

$$\zeta(s)\zeta(s-k) = \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^s}.$$

So: $\nu(n) = \sigma_0(n)$ and $\sigma(n) = \sigma_1(n)$.

Small computer project: Graph $\sigma_k(n)$ for $k = 0, 1, 2$.

4. The Moebius function $\mu(n)$ which is defined to be 0 if n is not square-free, and $+1$ if n is is a product of an even number of distinct primes[1], and is -1 if n is a product of an odd number of distinct primes.

Proposition 3 $\sum_{1 \le d \ | \ n} \mu(d) = 0$ *if* $n > 1$ *and is equal to* 1 *when* $n = 1$.

Proof: If $n = 1$ the proposition is obvious. If $n > 1$, n is divisible by some prime number p, so we can write $n = p^e \cdot n_o$ where $e = ord_p(n) > 1$ and where n_o is not divisible by p.

Note that

$$(*) \quad \mathrm{Div}(n) = \sqcup_{i=0}^{e} \ p^i \cdot \mathrm{Div}(n_o).$$

Put

$$\Delta := \sum_{d \in \mathrm{Div}(n_o)} \mu(d) = \sum_{1 \le d \ | \ n_o} \mu(d).$$

We *could* use induction to say that Δ is either zero or one, but we don't need to know this because:

- $\sum_{d \in p\mathrm{Div}(n_o)} \mu(d) = -\sum_{d \in \mathrm{Div}(n_o)} \mu(d) = -\Delta$, and
- $\sum_{d \in p^i\mathrm{Div}(n_o)} \mu(d) = 0$ if $i > 1$.

Therefore, by **(*)**,

$$\sum_{1 \le d \ | \ n} \mu(d) = \sum_{i=0}^{e} \sum_{d \in p^i \mathrm{Div}(n_o)} \mu(d) \ = \ \Delta \ - \ \Delta \ + \ 0 \ = \ 0.$$

[1]Note that zero is an *even* number.

Our text has a different proof of this. It comes by noting first that in the summation of the $\mu(d)$'s for divisors d of n the only nonzero contributions come from square-free d's and so you may as well assume that n itself is square-free, and a product of–say–r distinct primes ($r > 0$). Any such divisor is determined by stipulating the subset of those r primes that it is the product of. If it is a product of an odd (resp. even) number, then it contributes -1 (resp. $+1$) to the sum so you have

$$\sum_{1 \leq d \mid n} \mu(d) = 1 - \binom{r}{1} + \binom{r}{2} - \binom{r}{3} + \ldots + (-1)^r = (1-1)^r = 0.$$

The Moebius function defines an operator on functions of positive numbers, as follows: if $f(n)$ is such a function, define:

$$(Mf)(n) := \sum_{1 \leq d \mid n} \mu(d) f(n/d).$$

To think efficiently about this, we will introduce a certain kind of "multiplication of functions" that we've already encountered in the discussion above about Dirichlet series.

2 Dirichlet Product

Given functions $A : n \mapsto A(n)$ and $B : n \mapsto B(n)$, the formula

$$C(n) = \sum_{1 \leq d \mid n} A(d) \cdot B(n/d)$$

that we have encountered as giving the coefficients of the product of two Dirichlet Series defines a function $C(n)$ that can be thought of as a kind of product of A and B ("$\star$-multiplication"). Not quite following our text's notation, denote it:

$$C := A \star B.$$

This operation is commutative and associative, and there is even a two-sided identity $\mathbf{I}$.

$$(\mathbf{I} : n \mapsto 1 \text{ if } n = 1, \text{ and } 0 \text{ if } n > 1.)$$

In this new notation the ordinary product of two Dirichlet series $\sum_n A(n)/n^s$ and $\sum_n B(n)/n^s$ is

$$\sum_n A \star B(n)/n^s.$$

Also, note that our operation

$$f \mapsto Mf$$

is just $\star$-multiplying f by the Moebius function μ.

Proposition 4 *If ι is the constant function with value 1 (i.e., $\iota(n) = 1$ for all n) then*

$$M\iota = \mu \star \iota = \mathbf{I}.$$

That is, in terms of $\star$-product, the Moebius function μ is the $\star$-inverse of the constant function ι.

Corollary 2

$$\zeta(s)^{-1} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$

Proof: This follows from Proposition 3.

Moebius inversion:

Corollary 3 *For $n \mapsto F(n)$ an arbitrary function and form $f(n) := \sum_{1 \leq d \,|\, n} f(n/d)$. We can retrieve the original function F from f by*

$$F = Mf.$$

That is,

$$F(n) = \sum_{1 \leq d \,|\, n} \mu(d) f(n/d).$$

The proof consists in staring at the formula:

$$\iota \star \mu \star f = f.$$