Even more continued fractions and Pell's Equation.

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1 Reading Assignment:

Again Davenport: *All* of Chapter IV.

2 Recall: An application of 'reversal'

Let $\alpha \in \mathbf{R}$ be an irrational number whose *terms* in its continued fraction expansion given by the natural numbers

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

We have:

$$\alpha = \frac{[a_0, a_1, a_2, a_3, \ldots, a_n, \alpha_{n+1}]}{[a_1, a_2, a_3, \ldots, a_n, \alpha_{n+1}]}.$$

Recall:

$$[q_0, q_1, q_2, q_3, \ldots, q_n, q_{n+1}] = q_{n+1}[q_0, q_1, \ldots, q_n] + [q_0, q_1, \ldots, q_{n-1}].$$

So,

$$[a_0, a_1, a_2, a_3, \ldots, a_n, \alpha_{n+1}] = \alpha_{n+1}[a_0, a_1, \ldots, a_n] + [a_0, a_1, \ldots, a_{n-1}] =$$

$$= \alpha_{n+1}P_n + P_{n-1}.$$

and, of course:

$$[a_1, a_2, a_3, \ldots, a_n, \alpha_{n+1}] = \alpha_{n+1}[a_1, \ldots, a_n] + [a_1, \ldots, a_{n-1}] =$$
$$= \alpha_{n+1} Q_n + Q_{n-1}.$$

Therefore:

$$\alpha = \frac{[a_0, a_1, a_2, a_3, \ldots, a_n, \alpha_{n+1}]}{[a_1, a_2, a_3, \ldots, a_n, \alpha_{n+1}]} =$$

$$= \frac{\alpha_{n+1} P_n + P_{n-1}}{\alpha_{n+1} Q_n + Q_{n-1}}.$$

To record this:

Theorem 1

$$\alpha = \frac{\alpha_{n+1} P_n + P_{n-1}}{\alpha_{n+1} Q_n + Q_{n-1}}.$$

Corollary 2

$$|\alpha - \frac{P_n}{Q_n}| < \frac{1}{Q_n Q_{n+1}} < \frac{1}{Q_n^2}.$$

Proof: Compute:

$$\alpha - \frac{P_n}{Q_n} \;=\; \frac{\alpha_{n+1} P_n + P_{n-1}}{\alpha_{n+1} Q_n + Q_{n-1}} - \frac{P_n}{Q_n} =$$

$$= \frac{P_{n-1} Q_n - Q_{n-1} P_n}{Q_n(\alpha_{n+1} Q_n + Q_{n-1})} \;=\; \frac{\pm 1}{Q_n(\alpha_{n+1} Q_n + Q_{n-1})}.$$

But since $\alpha_{n+1} > a_{n+1}$ and $Q_{n+1} = a_{n+1} Q_n + Q_{n-1}$, we get:

$$|\alpha - \frac{P_n}{Q_n}| < \frac{1}{Q_n Q_{n+1}} < \frac{1}{Q_n^2}.$$

Discuss

$$|\alpha - X_k/Y_k| < \frac{1}{B \cdot Y_k^2}$$

for various values of B.

3 Purely periodic continued fractions

If

$$a_0, a_1, a_2, \ldots a_{n-1}, a_0, a_1, a_2, \ldots a_{n-1}, \ldots$$

are the terms of a continued fraction, we'll say that it is **purely periodic** and denote it

$$\overline{a_0,\ a_1,\ a_2,\ \ldots\ a_{n-1}}$$

and say that it has **period** n.

Theorem 3 ** Let D be a natural number, not a perfect square. Then $\sqrt{D} + \lfloor\sqrt{D}\rfloor$ has a purely periodic continued fraction expansion.*

So,

$$\sqrt{D} = \lfloor\sqrt{D}\rfloor;\ \overline{a_1,\ a_2,\ \ldots\ a_{n-1},\ 2\lfloor\sqrt{D}\rfloor}.$$

Or, more readably:

$$\sqrt{D} \;=\; \frac{1}{a_0+}\frac{1}{a_1+}\cdots\frac{1}{a_{n-1}+}\frac{1}{2a_0+}\frac{1}{a_1+}\cdots\frac{1}{a_{n-1}+}\frac{1}{2a_0+}\cdots,$$

with $a_0 = \lfloor\sqrt{D}\rfloor$.

3.1 Numerical examples

D	Continued Fraction for $\sqrt{D} + \lfloor\sqrt{D}\rfloor$
2	$\overline{2}$
3	$\overline{2,1}$
5	$\overline{4}$
7	$\overline{4,1,1,1}$
17	$\overline{8}$
26	$\overline{10}$
46	$\overline{12,1,3,1,1,2,6,2,1,1,3,1}$

3.2 Solving Pell's Equation

Corollary 4 *Let D be a natural number, not a perfect square. Let n be the period of the continued fraction expansion of $\alpha := \sqrt{D} + \lfloor\sqrt{D}\rfloor$. For any integer $k \geq 1$, put $m = kn - 1$ and consider P_{m-1}/Q_{m-1} and P_m/Q_m, the $m-1$-st and $m-th$ convergents of the continued fraction expansion of $\sqrt{D}$.*

Then

$$P^2 - DQ^2 = \pm 1.$$

Proof: P_{m-1}/Q_{m-1} and P_m/Q_m are the two convergents before the repeat of the continued fraction expansion of $\alpha = \sqrt{D} + \lfloor\sqrt{D}\rfloor$. Spo we have:

$$\sqrt{D} = \frac{\alpha P_m + P_{m-1}}{\alpha Q_m + Q_{m-1}}.$$

So,

$$\sqrt{D} \cdot (\sqrt{D} + \lfloor\sqrt{D}\rfloor)Q_m + Q_{m-1} = (\sqrt{D} + \lfloor\sqrt{D}\rfloor)P_m + P_{m-1}.$$

Now since $\sqrt{D}$ is irrational, and everything else is rational we get *two* equations out of this!

$$DQ_m = \lfloor\sqrt{D}\rfloor P_m + P_{m-1},$$

and

$$\lfloor\sqrt{D}\rfloor Q_m + Q_{m-1} = P_m.$$

So, expressing P_{m-1}, Q_{m-1} in terms of P_m, Q_m, we get:

$$P_{m-1} = DQ_m - \lfloor\sqrt{D}\rfloor P_m,$$

and

$$Q_{m-1} = P_m - \lfloor\sqrt{D}\rfloor Q_m.$$

Now, let's bring in the "gcd = 1" theorem:

$$P_m Q_{m-1} - Q_m P_{m-1} \;=\; \pm 1$$

to give (!!)

$$P_m^2 - DQ_m^2 = \pm 1.$$

3.3 Discuss open problems

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