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Anomalous eigenforms and the two-variable p-adic L-function.

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A p-ordinary p-adic modular (cuspidal) eigenform (for almost all Hecke operators T_ℓ with $\ell \neq p$ and for the Atkin-Lehner operator U_p) will be called **anomalous** if its U_p-eigenvalue is 1. There are four *kinds* of such anomalous eigenforms: Eisenstein series at p-adic weight κ where κ is a zero of a Leopold-Kubota p-adic L-function, certain classical p-new eigenforms of weight two, eigenforms associated to certain modular eigenforms of weight 1, and there is yet a further supply of non-classical anomalous eigenforms (at non-integral weights).

We shall revisit the theorem in [M-T-T] that proves that the p-adic L-function of an anomalous eigenform has a zero at every finite character χ such that $\chi(p) = 1$. We want to improve this result in the following way. First we wish to pin down a *canonical* two-variable p-adic L-function $L(\chi, \phi)$ where χ is the *character variable* and ϕ is the *eigenform variable*. In the body of this article, we will be thinking of ϕ as a ring-homomorphism of an appropriate Hecke algebra to a p-adic ring $\mathcal{O}$ of "Fourier coefficients." But in this introduction, let us simply think of ϕ as a parameter for the piece of the rigid analytic eigencurve corresponding to p-ordinary eigenforms with Fourier coefficients in $\mathbf{C}_p$. Fixing a finite $\mathbf{C}_p^*$-valued Dirichlet character χ such that $\chi(p) = 1$ we then will have

$$L_\chi(\phi) := L(\chi, \phi) \ \in \ \mathbf{C}_p,$$

where L_χ is a rigid analytic function on this piece of the eigencurve. The guaranteed zeroes of this rigid analytic function are at eigenforms of weight two when Birch and Swinnerton-Dyer predicts such a zero, and at anomalous eigenforms. It is interesting to ask whether there are any other zeroes.

Question 1: Are there any non-anomalous eigenforms ϕ of weight different from 2 such that $L_\chi(\phi) = 0$?

Clearly one cannot reasonably ask such a question until one has a moderately canonical construction of the two-variable p-adic L-function; i.e., a definition of the L-function that hasn't multiplied a better L-function by some spurious normalization function of ϕ that introduced extra, unnecessary zeroes at certain eigenforms ϕ.

We cannot yet answer the above question, but we can get to a point where, at least, it is possible to ask such a question, and refinements of it; we do this below (see **Question 2** at the end of section 3). But of course the first thing for us to get straight is to describe a *most canonical module* in which the two-variable p-adic L-function takes its values.

1. The p-adic "Symbol" modules $H^{\pm}_{\Delta} \subset \mathcal{H}^{\pm}_{\Delta}$. Let p be an odd prime number, and Δ a square-free positive integer prime to p, which in our eventual application I suppose we can take to be equal to 1. Let $\mathbf{Z}_{\Delta,p}$ denote the *ring*

$$\mathbf{Z}_{\Delta,p} := \text{proj.lim. } \mathbf{Z}/\Delta p^r\mathbf{Z}$$

(differing from Kitigawa) and I'll let $\mathbf{Z}^*_{\Delta,p}$ denote its ring of integers. Put

$$\mathbf{\Lambda}_{\Delta,p} = \mathbf{\Lambda}_{\Delta} := \mathbf{Z}_p[[\mathbf{Z}^*_{\Delta,p}]].$$

Form the affine, and the complete, modular curves $Y_1(\Delta p^r) \subset X_1(\Delta p^r)$ and let $C_1(\Delta p^r) := X_1(\Delta p^r) - Y_1(\Delta p^r)$ denote the cusps. All these may be considered as schemes over $\mathbf{C}$ until further notice. Consider the exact sequence of (standard "Betti") homology

$$0 \to H_1(X_1(\Delta p^r); \mathbf{Z}_p) \to H_1(X_1(\Delta p^r), C_1(\Delta p^r); \mathbf{Z}_p) \to H_0(C_1(\Delta p^r); \mathbf{Z}_p) \to \mathbf{Z}_p \to 0$$

noting that the Manin-Drinfeld splitting (using the fact that the Hecke eigenvaues of cusp forms and Eisenstein series are so cleanly differentiated) gives us a canonical projection

$$\pi : H_1(X_1(\Delta p^r), C_1(\Delta p^r); \mathbf{Z}_p) \to H_1(X_1(\Delta p^r); \mathbf{Q}_p)$$

such that π restricted to $H_1(X_1(\Delta p^r); \mathbf{Z}_p) \subset H_1(X_1(\Delta p^r), C_1(\Delta p^r); \mathbf{Z}_p)$ is the natural inclusion $H_1(X_1(\Delta p^r); \mathbf{Z}_p) \subset H_1(X_1(\Delta p^r); \mathbf{Q}_p)$.

Now the image module,

$$\pi\{H_1(X_1(\Delta p^r), C_1(\Delta p^r); \mathbf{Z}_p)\} \subset H_1(X_1(\Delta p^r); \mathbf{Q}_p)$$

is naturally a finitely generated $\mathbf{Z}_p[(\mathbf{Z}/\Delta p^r\mathbf{Z})^*]$-module via the action of the diamond operators; it is left stable by the action of the Hecke operators, and in particular by the Atkin-Lehner operator U_p.

Denote by $\mathbf{T}_1(\Delta p^r)$ the standard "p-ordinary" (i.e., *slope* 0) Hecke algebra: it is a $\mathbf{Z}_p$-algebra generated by the diamond operators, the Hecke operators T_ℓ for $\ell \not| \Delta p$ and by the U_q for the other prime numbers q, and by construction, $\mathbf{T}_1(\Delta p^r)$ acts faithfully on the modules $\mathcal{H}_1(X_1(\Delta p^r); \mathbf{Z}_p)^{\mathbf{ord}}$, the maximal quotient module of $\pi\{H_1(X_1(\Delta p^r), C_1(\Delta p^r); \mathbf{Z}_p)\}$ where U_p is invertible, and on its submodule

$$H_1(X_1(\Delta p^r); \mathbf{Z}_p)^{\mathbf{ord}} \subset \mathcal{H}_1(X_1(\Delta p^r); \mathbf{Z}_p)^{\mathbf{ord}}.$$

We may describe these modules as follows:

$$H_1(X_1(\Delta p^r); \mathbf{Z}_p)^{\mathbf{ord}} := H_1(X_1(\Delta p^r); \mathbf{Z}_p)\} \otimes_{\mathbf{Z}_p[U_p]} \mathbf{Z}_p[U_p, U_p^{-1}]$$

and

$$\mathcal{H}_1(X_1(\Delta p^r); \mathbf{Z}_p)^{\mathbf{ord}} := \pi\{H_1(X_1(\Delta p^r), C_1(\Delta p^r); \mathbf{Z}_p)\} \otimes_{\mathbf{Z}_p[U_p]} \mathbf{Z}_p[U_p, U_p^{-1}].$$

Pass to the projective limit letting r tend to ∞.

Definitions.

$$\mathbf{T}_{\Delta,p} = \mathbf{T}_\Delta := \text{proj.lim. } \mathbf{T}_1(\Delta p^r),$$
$$H_{\Delta,p} = H_\Delta := \text{proj.lim. } H_1(X_1(\Delta p^r); \mathbf{Z}_p)^{\mathbf{ord}},$$
$$\mathcal{H}_{\Delta,p} = \mathcal{H}_\Delta := \text{proj.lim. } \mathcal{H}_1(X_1(\Delta p^r); \mathbf{Z}_p)^{\mathbf{ord}}.$$

The $\mathbf{\Lambda}_\Delta$-algebra $\mathbf{T}_\Delta$ is finite flat (reference?). We have an inclusion of (faithful) $\mathbf{T}_\Delta$-modules $H_\Delta \subset \mathcal{H}_\Delta$ and each of these modules (after tensoring with $\mathbf{Q}_p$) are of rank two over $\mathbf{T}_\Delta \otimes \mathbf{Q}_p$. To figure out exactly what is happening here, we should look more closely at the question of freeness or non-freeness of H_Δ and $\mathcal{H}_\Delta$ over $\mathbf{T}_\Delta$ but I'll leave that for later.

Complex conjugation commutes with the action of $\mathbf{T}_\Delta$. Taking symmetric and anti-symmetric "averages" we may split the inclusion $H_\Delta \subset \mathcal{H}_\Delta$ into a direct sum of two inclusions of $\mathbf{T}_\Delta$-modules, $H_\Delta^\pm \subset \mathcal{H}_\Delta^\pm$, these latter modules being of "rank one" (meaning that after tensoring with $\mathbf{Q}_p$ they are of rank one over $\mathbf{T}_\Delta \otimes \mathbf{Q}_p$). For a finer description of these $\mathbf{T}_\Delta$-modules see sections 3-5 of [M-W]. Note that in that reference we chuck out the eigenspace connected to the trivial tame character, so one should make a separate analysis of that case.

2. The p-adic "Modular Symbol." For every rational number a/b the oriented 1-chain going from the cusp ∞ to a/b via a straight line in the upper half-plane projects to an oriented 1-chain going from the image of the cusp $\infty \in X_1(\Delta p^r)$ to the image of the cusp $a/b \in X_1(\Delta p^r)$ and therefore represents a homology class $h_1(\Delta p^r)\{a/b\} \in H_1(X_1(\Delta p^r), C_1(\Delta p^r); \mathbf{Z}_p)$. Put

$$\xi_1(\Delta p^r)\{a/b\} := \text{image of } \pi h_1(\Delta p^r)\{a/b\} \text{ in } \mathcal{H}_1(X_1(\Delta p^r); \mathbf{Z}_p).$$

These classes compile well under the mappings

$$X_1(\Delta p^{r+1}) \to X_1(\Delta p^r)$$

and they produce, in the projective limit, classes

$$\xi_\Delta^\pm\{a/b\} := \text{proj.lim. } \xi_1(\Delta p^r)\{a/b\}^\pm \quad \in \quad \mathcal{H}_\Delta^\pm.$$

The value of $\xi^{\pm}_{\Delta}$ depends only on the rational number a/b modulo 1 and therefore we get a well-defined mapping

$$\xi^{\pm}_{\Delta} : \mathbf{Q}/\mathbf{Z} \to \mathcal{H}^{\pm}_{\Delta}$$

which gives us $\mathcal{H}^{\pm}_{\Delta}$-valued measures, as usual, on the topological group $\mathbf{Z}_{\Delta,p}$. Explicitly, for integers a prime to Δp, and for integers $\nu \geq 1$ define :

$$\mu^{\pm}_{\Delta}(a + \Delta p^{\nu}\mathbf{Z}_{\Delta}) := U_p^{-\nu} \cdot \xi^{\pm}_{\Delta}\{a/\Delta p^{\nu}\} \quad \in \quad \mathcal{H}^{\pm}_{\Delta}.$$

The open subsets $a + \Delta p^{\nu}\mathbf{Z}_{\Delta} \subset \mathbf{Z}^{*}_{\Delta}$ form a base for the topology and a direct calculation shows that $\mu^{\pm}_{\Delta}$ defined as above extends to yield a finitely additive measure on open subsets of $\mathbf{Z}^{*}_{\Delta}$ with values in $\mathcal{H}^{\pm}_{\Delta}$. The p-adic Mellin transform of this measure, say for $\mathbf{Z}^{*}_{p}$-valued characters χ,

$$\chi \quad \mapsto \quad L_{\chi} := \int_{\mathbf{Z}^{*}_{\Delta}} \chi \mu^{\mathrm{sgn}(\chi)}_{\Delta} \in \mathcal{H}^{\pm}_{\Delta}$$

"is" the canonical two-variable p-adic L-function. To generalize our range of characters and be more *two-variable-ish,* let us consider a p-complete $\mathbf{Z}_p$-algebra $\mathcal{O}$ together with some pair (χ, ϕ) where

$$\chi : \mathbf{Z}^{*}_{\Delta} \to \mathcal{O}^{*}$$

is a continuous character and

$$\phi : \mathbf{T}_{\Delta} \to \mathcal{O}$$

is a ring-homomorphism (any such ϕ determines and is determined by a p-ordinary eigenform of tame level Δ and with Fourier coefficients in the ring $\mathcal{O}$). Denote by the same letter ϕ the induced homomorphism

$$\phi : \mathcal{H}^{\pm}_{\Delta} \to \mathcal{O} \otimes_{\mathbf{T}_{\Delta}} \mathcal{H}^{\pm}_{\Delta}$$

(the tensor product being via ϕ). Then we can define the value of the p-adic L-function on the pair (χ, ϕ) as follows:

$$L(\chi, \phi) := \phi\{\int_{\mathbf{Z}^{*}_{\Delta}} \chi \mu^{\mathrm{sgn}(\chi)}_{\Delta}\} \in \mathcal{O} \otimes_{\mathbf{T}_{\Delta}} \mathcal{H}^{\mathrm{sgn}(\chi)}_{\Delta}.$$

We might refer to χ as the *character variable* and ϕ as the *eigenform variable.*

3. Evaluation of the two-variable p-adic L-function on characters of conductor prime to p. Let $\chi : (\mathbf{Z}/\Delta\mathbf{Z})^{*} \to \mathcal{O}^{*}$ be such a character, which we also view as a character on $\mathbf{Z}^{*}_{\Delta}$. Put

$$A_{\chi} := \sum_{j \in \mathbf{Z}^{*}_{\Delta}} \chi^{-1}(j)\xi^{\mathrm{sgn}(\chi)}_{\Delta}(j/\Delta) \in \mathcal{O} \otimes_{\mathbf{Z}_p} \mathcal{H}^{\mathrm{sgn}(\chi)}_{\Delta}.$$

Proposition.

$$L_\chi \;=\; (1-\chi^{-1}(p)U_p)\cdot A_\chi \quad \in \quad (1-\chi^{-1}(p)U_p)\cdot \mathcal{O}\otimes_{\mathbf{Z}_p}\mathcal{H}_\Delta^{\mathrm{sgn}(\chi)}.$$

Proof. This repeats either the proof of the analogous formula in [M-T-T] or the same calculation is given in Lemma 4.5 of Kitigawa's article.

Definition. A p-ordinary eigenform of tame level Δ and with Fourier coefficients in the ring $\mathcal{O}$, given by a ring-homomorphism $\phi : \mathbf{T}_\Delta \to \mathcal{O}$ is called **anomalous** if

$$\phi(U_p) = 1 \quad \in \quad \mathcal{O}.$$

Corollary. Let $\chi : (\mathbf{Z}/\Delta\mathbf{Z})^* \to \mathcal{O}^*$ be a character such that $\chi(p) = 1$. Then

$$L_\chi \quad \in \quad \mathcal{O}\otimes_{\mathbf{Z}_p} H_\Delta^{\mathrm{sgn}(\chi)} \;\subset\; \mathcal{H}_\Delta^{\mathrm{sgn}(\chi)}.$$

If $\phi : \mathbf{T}_\Delta \to \mathcal{O}$ is the ring-homomorphism attached to an anomalous eigenform, then

$$L(\chi,\phi) \quad = \quad 0.$$

Proof. The first statement follows since

$$(1-U_p)\cdot\mathcal{H}_\Delta^{\mathrm{sgn}(\chi)} \;\subset H_\Delta^{\mathrm{sgn}(\chi)}.$$

The second is a direct corollary of the proposition.

Question 2: Is the annihilator ideal in $\mathbf{T}_\Delta$ of the $\mathbf{T}_\Delta$-module

$$\mathcal{O}\otimes_{\mathbf{Z}_p}\mathcal{H}_\Delta^{\mathrm{sgn}(\chi)}/A_\chi\cdot\mathbf{T}_\Delta$$

contained in the weight two kernel of $\mathbf{T}_\Delta$?

4. References.

[K] Kitigawa, K.: *On p-adic L-functions of families of cusp forms,* in *p-adic Monodromy and the Birch and Swinnerton-Dyer Conjecture,* (eds.: B. Mazur, G. Stevens) Contemporary Mathematics Series **165**, A.M.S. (1994).

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