Introductory comments on the eigencurve

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Handout # 2.

1 Classical Eisenstein series

Let p be an odd prime number. Recall the classical family of Eisenstein series E_k for k an even integer (> 2)

$$E_k = 1 - \frac{2k}{b_k} \sum_{n \geq 1} \sigma_{k-1}(n) q^n,$$

where b_k is the k-th Bernoulli number and

$$\sigma_{k-1}(n) := \sum_{0 < d \,|\, n} d^{k-1},$$

so that

$$E_4 = 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n,$$

and

$$E_6 = 1 - 504 \sum_{n \geq 1} \sigma_5(n) q^n,$$

and

$$E_{10} = E_4 E_6 = 1 - 11 \cdot 24 \sum_{n \geq 1} \sigma_9(n) q^n \equiv 1 \bmod 11,$$

and more generally

$$E_k \equiv 1 \bmod 11$$

for k divisible by 10

These power series E_k are the Fourier series of the most basic of Eisenstein series, and are modular eigenforms (for all Hecke operators) of weight k and of level one. This family doesn't yet p-adically interpolate. We need to augment these forms to be eigenforms for $\Gamma_0(p)$.

Put, then,

$$E_k^{(p)}(q) = E_k(q) - p^{k-1}E_k(q^p)$$

giving us eigenforms on $\Gamma_0(p)$ with Fourier expansions

$$E_k^{(p)}(q) = 1 - p^{k-1} - \frac{2k}{b_k} \sum_{n \geq 1} \sigma_{k-1}^{(p)}(n)q^n,$$

where

$$\sigma_{k-1}^{(p)}(n)q^n := \sum_{0 < d \mid n; \ (p,d)=1} d^{k-1}.$$

There are *two* fairly natural ways of renormalizing these Eisenstein series, depending upon the congruence class of $k \bmod p - 1$.

- For $k \not\equiv 0 \bmod p - 1$ divide by $-\frac{2k}{b_k}$ to get:

$$\mathbf{Eis}_k^{(p)}(q) := -(1 - p^{k-1})\frac{b_k}{2k} + \sum_{n \geq 1} \sigma_{k-1}^{(p)}(n)q^n,$$

- In the specific arithmetic progression $k \geq 2$, $k \equiv 0$ modulo $p - 1$, it is better to renormalize $E_k^{(p)}(q)$ by dividing by $1 - p^{k-1}$ to get:

$$\mathcal{E}_k^{(p)}(q) = 1 - \frac{2k}{(1 - p^{k-1})b_k} \sum_{n \geq 1} \sigma_{k-1}^{(p)}(n)q^n,$$

Both of these—thanks to the Kummer congruences, and Euler theorem— interpolate p-adically, and come from Λ-adic Eisenstein series as previously discussed.

Specifically, let
$$W := \lim_{\nu} 2\mathbf{Z}/(p-1)p^{\nu-1}\mathbf{Z} = 2\mathbf{Z}/(p-1)\mathbf{Z} \times \mathbf{Z}_p.$$

The even positive numbers, $2\mathbf{N}$, form a dense subset of W. We view W as a *space of p-adic weights*. Later we will think of W as the *classically accessible weights* in the rigid-analytic space $\mathcal{W}$ of $\mathbf{C}_p$-valued weights (a union of $\frac{p-1}{2}$ disjoint open discs). But for the moment, put

$$W = W_0 \sqcup W_2 \sqcup \ldots \sqcup W_{p-2}$$

where $W_i = \{w \in W \mid w \equiv i \mod p-1\}$. So each W_i may be canonically identified with $\mathbf{Z}_p$. For each such i put

For $i \in 2\mathbf{Z}/(p-1)\mathbf{Z}$ define $\sigma_{\{i\}}(n) \in \Lambda$ to be

$$\sum_{0 < d \mid n; \ (p,d)=1} \omega^i(d)\langle d \rangle \in \Lambda.$$

Also,

$$L_p(1 - k, \omega^k) = -(1 - p^{k-1})\frac{b_k}{2k},$$

(cf. Larry Washington's *Introduction to Cyclotomic Fields*, and in particular his treatment of Kummer congruences (Corollary 5.14), page 61).

Here ω is the Teichmüller character, $i \in 2\mathbf{Z}/(p-1)\mathbf{Z}$, and $L_p(s, \omega^i)$ is the Leopoldt-Kubota p-adic L-function.

When $i \not\equiv 0 \mod p-1$, $L_p(s, \omega^i)$ comes from an "Iwasawa function" in $L_{p,\{i\}} \in \Lambda$ under specialization to weight "$1 - s$."

When $i \equiv 0 \mod p-1$ it is rather the *inverse*, $L_p(s, \omega^i)^{-1}$ that comes from an "Iwasawa function" that we will denote $L_{p,\{0\}}^{-1} \in \Lambda$ under specialization to weight "$1 - s$." Moreover, by Clausen- Von Staudt,

$$L_{p,\{0\}}^{-1} \equiv 1 \mod p.$$

Our two Eisenstein series then come from Λ-adic counterparts,

- For $i \not\equiv 0$,

$$\mathbf{Eis}_{\{i\}}^{(p)}(q) := L_{p,\{i\}} + \sum_{n \geq 1} \sigma_{\{i\}}^{(p)}(n)q^n \in \Lambda[[q]]$$

- For $k \equiv 0$ modulo $p-1$, we have:

$$\mathcal{E}_{\{0\}}^{(p)}(q) = 1 - L_{p,\{0\}}^{-1} \sum_{n \geq 1} \sigma_{\{0\}}^{(p)}(n)q^n \in \Lambda[[q]].$$

2 On the uses of the *Eisenstein lines* $\mathbf{Eis}_{\{i\}}^{(p)}$ and $\mathcal{E}_{\{0\}}^{(p)}$.

The main use of $\mathcal{E}_{\{0\}}^{(p)}$ stems from the fact that

$$\mathcal{E}^{(p)}_{\{0\}} \equiv 1 \bmod p,$$

its specialization to weight 0 is the constant function 1, and finally its specialization to an even positive weight is a classical modular (eigen-) form of that weight. Therefore $\mathcal{E}^{(p)}_{\{0\}}$ can help to spread out a classical modular form f of a given weight k, by straightforward multiplication

$$f \cdot \mathcal{E}^{(p)}_{\{0\}} \in \Lambda[[q]]$$

to give us a Λ-adic form with the property that

- Its Fourier series taken modulo p is *constant*, and more specifically

$$f \cdot \mathcal{E}^{(p)}_{\{0\}} \equiv f \bmod p,$$

- $f \cdot \mathcal{E}^{(p)}_{\{0\}}$ specializes to the given form f in weight k,

- $f \cdot \mathcal{E}^{(p)}_{\{0\}}$ specializes to a classical modular form in even (integral) weights $k' \geq k$ such that $k' \equiv k \bmod p - 1$.

Of course, even if f was a modular eigenform, the specialized classical modular forms in weights $k' \geq k$ such that $k' \equiv k \bmod p - 1$ may not be; but we will deal with this soon.

The *Eisenstein lines* $\mathbf{Eis}^{(p)}_{\{i\}}(q)$ play a different role. Here we can illustrate it with the first interesting case $p = 37$ where, when $i = 32$ the p-adic L function has a zero (a unique zero, in fact) and therefore the Fourier series of $\mathbf{Eis}^{(37)}_{\{32\}}(q)$ when specialized to the weight corresponding to that zero (probably a transcendental 37-adic weight) we get the Fourier series of something that "looks like" a cuspform. Discuss this.

3   Getting the cuspidal Λ-adic family from the Eisenstein family

Now return to the set-up of our previous handout, with $p = 11$. We may also view our cuspidal Λ-adic Fourier expansion,

$$F = \sum_{n \geq 0} T_n q^n \in \Lambda[[q]],$$

of the previous handout, as a family parametrized by W_2, setting

$$F_\kappa = \sum_{n \geq 0} \kappa(T_n) q^n \in \mathbf{Z}_p[[q]]$$

for $\kappa \in W$. Here, again, if κ is an even positive integer k congruent to 2 mod $p - 1$ then F_κ is a classical modular cuspidal eigenform on $\Gamma_0(p)$ of weight k.

Theorem 1 *(Construction of $F = \sum_{n \geq 0} T_n q^n \in \Lambda[[q]]$)*

The limit

$$\mathcal{F} := \lim_{\nu \to \infty} Normalize\{U_p^\nu(f_2 \cdot \mathcal{E}^{(p)})\} \in \Lambda[[q]]$$

exists, and has the property that for every even positive integer k congruent to $2 \bmod p - 1$ the specialization $\mathcal{F}_k$ of $\mathcal{F}$ to weight k is equal to the unique normalized cuspidal eigenform f_k with p-adic unit U_p-eigenvalue. [So, $\mathcal{F}$ is the Λ-adic form denoted F in the previous handout.]

Discussion of the ingredients of a proof of this.

Consider any infinite sequence of distinct weights $w_j : \Lambda \to \mathbf{Z}_p$ for $j = 1, 2, 3, \dots$ and let

$$I_m := \cap_{j=1,2,\dots m} \ker(w_j) \subset \Lambda$$

so that we get a decreasing sequence of ideals

$$I_1 \supset I_2 \supset \dots$$

with

$$\cap_{m=1}^\infty I_m = \{0\}.$$

Let

$$F_{\nu,N} := Normalize\{U_p^\nu(f_2 \cdot \mathcal{E}^{(p)})\} \text{ modulo } p^N,$$

and we will show first that this sequence, for fixed N and ν tending to infinity achieves a limit.

The main *real* ingredient of the proof of this is just that the U_p-unit eigensubspace of the space $S_k(\Gamma_0(11))$ is of $\mathbf{Z}_p$-rank 1, generated by a (unique, therefore) normalized eigenform (denoted f_k in the previous handout) for $k \geq 2$ with $k \equiv 2 \bmod p - 1$. We will be taking our weights $w_j := 2 + j(p-1)$.

[Recall, in this section, $p = 11$ but we call it p so as not to be typographically inundated with 11's.]

For any N, the intersection of the ideals (I_m, p^N) for all m is equal to the ideal generated by p^N in Λ, and for any even positive integer k congruent to $2 \bmod p - 1$, there exists an integer $\nu(k, N)$ such that if $\nu \geq \nu(k, N)$ then

$$U_p^\nu(S_k(\Gamma_0(p), \mathbf{Z}_p)) \subset f_k \cdot \mathbf{Z}_p + p^N S_k(\Gamma_0(p), \mathbf{Z}_p).$$

So, if ν, ν' are larger than $\max\{\nu(w_j, N); j = 1, \dots m\}$ then

$$F_{\nu,N} \equiv F_{\nu',N} \bmod (I_m, p^N),$$

and we get that

$$F_{\infty,N} := \lim_{\nu \to \infty} F_{\nu,N} \in \Lambda/p^N \Lambda[[q]]$$

exists, and moreover they compile well for varying N giving us an element

$$\mathcal{F} = \lim_{\nu \to \infty} F_{\infty,N} \in \Lambda[[q]]$$

which is easily seen to satisfy the statement of the theorem.

4   Hida's Theory

Discuss the general implications of this (the *principle of local constancy of U_p-unit eigenspace rank*). State Hida's theorem in slightly less generality than maximal. For this, fix $p \geq 5$ a prime number and (a tame level) N a positive integer prime to p. Consider the tower of modular curves

$$\ldots \rightarrow Y_1(Np^{r+1}) \rightarrow Y_1(Np^r) \rightarrow \ldots \rightarrow Y_1(Np)$$

which we will think of first as Riemann surfaces and later as algebraic curves defined over $\mathbf{Q}$. We can take (first: standard singular) homology with $\mathbf{Z}_p$ coefficients:

$$H_1(Y_1(Np^r), \mathbf{Z}_p) = \Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbf{Z}_p.$$

Define the intermediate groups

$$\Phi_r := \Gamma_1(Np) \cap \Gamma_0(p^r)$$

so we have

$$\Gamma_1(Np^r) \subset \Phi_r \subset \Gamma_1(Np)$$

noting that Φ_r is the normalizer of $\Gamma_1(Np^r)$ in $\Gamma_1(Np)$.

Recall that Γ (undecorated) is the group of 1-units in $\mathbf{Z}_p^*$ and let $\Gamma_r \subset \Gamma$ be the subgroup of index p^r. Define a surjective homomorphism of groups

$$\Phi_r \rightarrow \Gamma/\Gamma_r$$

by the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow d \bmod p^r$$

giving us the exact sequence

$$1 \rightarrow \Gamma_1(Np^r) \rightarrow \Phi_r \rightarrow \Gamma/\Gamma_r \rightarrow 1,$$

which, in turn, gives us an action of Γ/Γ_r on $\Gamma_1(Np^r)^{\mathrm{ab}}$ by conjugation, this being the *diamond action*. This extends, then, to give us a Λ-module structure $\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbf{Z}_p$ and from this, passing to the limit we may consider $\mathbf{H} := \lim_r H_1(Y_1(Np^r), \mathbf{Z}_p)$ as Λ-module.

The action of U_p is the correspondence induced by the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$$

via the following piece of "group theory." Since $U^{-1}\Gamma_1(Np^r)U \cap \Gamma_1(Np^r)$ is of finite index in $\Gamma_1(Np^r)$ we have homomorphisms,

$$\Gamma_1(Np^r)^{\mathrm{ab}} \to \{U^{-1}\Gamma_1(Np^r)U \cap \Gamma_1(Np^r)\}^{\mathrm{ab}} \to \{\Gamma_1(Np^r) \cap U^{-1}\Gamma_1(Np^r)\}^{\mathrm{ab}} \to \Gamma_1(Np^r)^{\mathrm{ab}},$$

whose composition is the Atkin-lehner operator U_p.

In what follows below, the superscript "ord" will mean that we have passed to the maximal sub-module on which U_p acts as a unit; equivalently, we have applied Hida's projection operator.

Let $\mathbf{T}^{\mathrm{ord}} \subset \mathrm{End}_\Lambda(\mathbf{H}^{\mathrm{ord}}$ be the Λ-subalgebra generated by all the Hecke operators T_ℓ for ℓ not dividing pN and the $U - q$'s for prime numbers q dividing pN, noting that U_p acts here as a unit, and by Δ, the group of tame diamond operators mod p.

Theorem 2 *The Λ-module*
$$\mathbf{H}^{\mathrm{ord}} := \lim_{r} \ H_1(Y_1(Np^r), \mathbf{Z}_p)^{\mathrm{ord}}$$
is a free Λ-module of finite rank, and a perfect control theorem holds; that is, for all $r \geq 1$
$$\mathbf{H}^{\mathrm{ord}} \otimes_\Lambda \mathbf{Z}_p[\Gamma/\Gamma_r] \cong H_1(Y_1(Np^r), \mathbf{Z}_p)^{\mathrm{ord}}.$$

To avoid some very minor technical issues, let me state the next theorem in a special case.

Theorem 3 *Let $N = 1$ and $m \subset \mathbf{T}^{\mathrm{ord}}$ a maximal ideal, that is not an Eisenstein[1] maximal ideal. Let the subscript m denote completion with respect to m. Then the $\mathbf{T}^{\mathrm{ord}}_m$-module*
$$\mathbf{H}^{\mathrm{ord}}_m = \lim_{r} \ H_1(Y_1(p^r), \mathbf{Z}_p) \otimes_{\mathbf{T}} \mathbf{T}^{\mathrm{ord}}$$
is a free $\mathbf{T}^{\mathrm{ord}}_m$-module of rank two.

Discuss Eisenstein part, and cuspidal part, of this. Discuss action of Galois, and the representations

$$G_{\mathbf{Q},\{p,\infty\}} \longrightarrow \mathbf{GL}_2(\mathbf{T}^{\mathrm{ord}}_m).$$

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[1]This will be defined later: it is going to be equivalent to the statement that the residual representation is not reducible.

References:

1. M. Emerton: A new proof of a theorem of Hida, IMRN, **9** (1999) 453-472.

2. M. Emerton: The Eisenstein ideal in Hida's Ordinary Hecke Algebra, IMRN **15** (1999) 793-802.

3. H. Hida: Galois representations into $\mathrm{GL}_2(\mathbf{Z}_p[[X]])$ attached to ordinary cusp forms, Invent. math. **85** (1986) 545-613.

4. H. Hida: p-Adic ordinary Hecke algebras for $\mathrm{GL}(2)$, Ann. Inst. Fourier, Grenoble **44** (1994) 1289-1322.