Introductory comments on the eigencurve

February 28, 2006

Handout # 4.

1 Hida's Theory

Discuss the general implications of this (the *principle of local constancy of U_p-unit eigenspace rank*). State Hida's theorem in slightly less generality than maximal ([14], [15]; [7], [8]). For this, fix $p \geq 5$ a prime number and (a tame level) N a positive integer prime to p. Consider the tower of modular curves

$$\ldots \to Y_1(Np^{r+1}) \to Y_1(Np^r) \to \ldots \to Y_1(Np)$$

which we will think of first as Riemann surfaces and later as algebraic curves defined over $\mathbf{Q}$. We can take (first: standard singular) homology with $\mathbf{Z}_p$ coefficients:

$$H_1(Y_1(Np^r), \mathbf{Z}_p) = \Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbf{Z}_p.$$

Define the following intermediate groups.

- For $r \geq 1$ define

  $$\Phi_r := \Gamma_1(Np) \cap \Gamma_0(p^r)$$

  so we have

  $$\Gamma_1(Np^r) \subset \Phi_r \subset \Gamma_1(Np)$$

  noting that Φ_r is the normalizer of $\Gamma_1(Np^r)$ in $\Gamma_1(Np)$.

- For $r \geq s \geq 1$ define

  $$\Phi_{r,s} := \Gamma_1(Np^s) \cap \Gamma_0(p^r).$$

Recall that Γ (undecorated) is the group of 1-units in $\mathbf{Z}_p^*$ and let $\Gamma_r \subset \Gamma$ be the subgroup of index p^r. Define a surjective homomorphism of groups

$$\Phi_r \to \Gamma/\Gamma_r$$

by the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \ d \bmod p^r$$

giving us the exact sequence

$$1 \to \Gamma_1(Np^r) \to \Phi_r \to \Gamma/\Gamma_r \to 1,$$

which, in turn, gives us an action of Γ/Γ_r on $\Gamma_1(Np^r)^{\mathrm{ab}}$ by conjugation, this being the *diamond action.* This extends, then, to give us a Λ-module structure $\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbf{Z}_p$ and from this, passing to the limit we may consider $\mathbf{H} := \lim_r \ H_1(Y_1(Np^r), \mathbf{Z}_p)$ as Λ-module.

The action of U_p is the correspondence induced by the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$$

via the following piece of "group theory." Since $U^{-1}\Gamma_1(Np^r)U \cap \Gamma_1(Np^r)$ is of finite index in $\Gamma_1(Np^r)$ we have homomorphisms,

$$\Gamma_1(Np^r)^{\mathrm{ab}} \to \{U^{-1}\Gamma_1(Np^r)U \cap \Gamma_1(Np^r)\}^{\mathrm{ab}} \to \{\Gamma_1(Np^r) \cap U^{-1}\Gamma_1(Np^r)\}^{\mathrm{ab}} \to \Gamma_1(Np^r)^{\mathrm{ab}},$$

whose composition is the Atkin-Lehner operator U_p.

In what follows below, the superscript "ord" will mean that we have passed to the maximal submodule on which U_p acts as a unit; equivalently, we have applied Hida's projection operator.

Let $\mathbf{T}^{\mathrm{ord}} \subset \mathrm{End}_\Lambda(\mathbf{H}^{\mathrm{ord}})$ be the Λ-subalgebra generated by all the Hecke operators T_ℓ for ℓ not dividing pN and the U_q's for prime numbers q dividing pN, noting that U_p acts here as a unit, and by $\Delta = \Delta_{pN}$, the group of tame diamond operators mod pN.

Theorem 1 *The Λ-module*

$$\mathbf{H}^{\mathrm{ord}} := \lim_r \ H_1(Y_1(Np^r), \mathbf{Z}_p)^{\mathrm{ord}}$$

is a free Λ-module of finite rank, and a perfect control theorem holds; that is, for all $r \geq 1$

$$\mathbf{H}^{\mathrm{ord}} \otimes_\Lambda \mathbf{Z}_p[\Gamma/\Gamma_r] \cong H_1(Y_1(Np^r), \mathbf{Z}_p)^{\mathrm{ord}}.$$

The essential piece in the proof of this theorem is a control lemma at finite layers. Specifically, for any positive integer r define the ideal

$$a_r := \ker\{\Lambda \to \Lambda_r := \mathbf{Z}_p[\Gamma/\Gamma_r]\}.$$

Lemma. For $s \leq r$ the natural mappings

$$(\Gamma_1(Np^r)^{\mathrm{ab}} \otimes \mathbf{Z}_p)^{\mathrm{ord}}/a_s \to (\Phi_{r,s}^{\mathrm{ab}} \otimes \mathbf{Z}_p)^{\mathrm{ord}} \to (\Gamma_1(Np^s)^{\mathrm{ab}} \otimes \mathbf{Z}_p)^{\mathrm{ord}}$$

are both isomorphisms.

So, the composition of these homomorphisms, which is a "control homomorphism," is also an isomorphism. Discuss how to get the full theorem from this. Also discuss alternate routes connected to the cohomology of the tower of Igusa curves (cf. [20]). For more on this, see *Anomalous eigenforms and the two-variable p-adic L-function* on the web-page of our course.

To avoid some very minor technical issues, let me state the next theorem in a special case where $N = 1$ but those same technical issues can be avoided if we work with $\Gamma_0(N) \cup \Gamma_0(p^r)$ and keep to N square-free.

Theorem 2 *Let $N = 1$ and $m \subset \mathbf{T}^{\mathrm{ord}}$ a maximal ideal, that is not an Eisenstein*[1] *maximal ideal. Let the subscript m denote completion with respect to m. Then the Λ-algebra $\mathbf{T}_m^{\mathrm{ord}}$ is finite and flat, and the $\mathbf{T}_m^{\mathrm{ord}}$-module*

$$\mathbf{H}_m^{\mathrm{ord}} = \lim_{r}\ H_1(Y_1(p^r), \mathbf{Z}_p) \otimes_{\mathbf{T}} \mathbf{T}^{\mathrm{ord}}$$

is a free $\mathbf{T}_m^{\mathrm{ord}}$-module of rank two.

Computational Query: (Keep to $N = 1$ and m non-Eisenstein.) The rank of $\mathbf{T}_m^{\mathrm{ord}}$ over Λ is the $\mathbf{Z}_p$-rank of $H^1(X_1(p); \mathbf{Z}_p)_m^{\mathrm{ord}}$. Call this rank $r_1(p)_m^{\mathrm{ord}}$. This is (essentially) computable from the Newton polygon of the U_p-operator acting on cuspforms of weight k on $\Gamma_0(p)$, for which we have, nowadays, good programs. It might be interesting to make a systematic computation, in hopes of getting a more general hold on the structure of the Λ-algebra $\mathbf{T}_m^{\mathrm{ord}}$. For example, the discriminant of $\mathbf{T}_m^{\mathrm{ord}}$ over Λ plays an important role in our theory, and it would be good to have lots of data about this. The underlying geometric question here is whether (and if so, where) the natural projection mapping of the rigid analytic space attached to $\mathrm{Spec}(\mathbf{T}_m^{\mathrm{ord}})$ is ramified over weight space. That these points of ramification over weight space (when they exist) correspond to certain zero-values of a p-adic L-function attached to the symmetric square is the subject of a current Berkeley PhD thesis of Walter Kim.

For our initial motivating example ($p = 11$ and m corresponding to $\{i\} \equiv 2 \bmod p - 1 = 10$) we have $r_1(p)_m^{\mathrm{ord}} = 1$ and therefore $\Lambda = \mathbf{T}_m^{\mathrm{ord}}$.

[1]This will be defined later: it is going to be equivalent to the statement that the residual representation is not reducible.

Table 1: ordinary rank of $S_k(\Gamma_0(p)$ for p $= 37$ and $k \equiv i \bmod 36$, k even, and $k \geq 2$

$\{i\}$	$rank$
0	3
2	2
$4 \leq \{i\} \leq 10$	0
12	1
16	1
18	1
20	1
22	1
24	2
26	1
28	2
30	2
32	2
34	2

A similar situation holds for $p = 13$ where, again the modular form Δ of weight 12 provides the only ordinary eigenform (this time, of course, we have $\{i\} \equiv 0 \bmod p - 1 = 12$).

When $p = 17$ we have single ordinary eigenforms of weights $12, 16, 18$ and so the corresponding Λ-algebras $\mathbf{T}_m^{\mathrm{ord}}$'s are each isomorphic to Λ.

The same goes for $p = 19$ where we have single ordinary eigenforms of weights $12, 16, 18$, and 20.

Things become interesting for $p = 23$, where there are single ordinary eigenforms of weights $12, 16, 18, 20$, and 22, but there are two eigenforms of weight 2 (and they are attached to the same maximal ideal m). So the corresponding Hecke algebra $\mathbf{T}_m^{\mathrm{ord}}$ is quadratic over Λ (I believe that it is unramified ...). We have, similarly, some quadratic $\mathbf{T}_m^{\mathrm{ord}}$'s for $p = 29$ and $p = 31$.

The fun, of course, is to find examples with $r_1(p)_m^{\mathrm{ord}} > 1$ and to try to understand the structure of the corresponding Λ-algebra $\mathbf{T}_m^{\mathrm{ord}}$. Some data can be gotten from Fernando Gouvêa's article "where the slopes are" (See his web-page, and more specifically, see the tabulated results

`http://http://www.colby.edu/personal/fqgouvea/slopes/).`

The table at the top of this page takes a bit of information from that source (that tabulates, in effect, the Newton polygon of the U_p-operator acting on $S_k(\Gamma_0(p)$ for a range of k and p).

Here is a result which we will be examining later, where we are momentarily extending our range to include N square-free (cf. [17] for a complete proof of this; the sketch we give below contains the main ingredients of the proof).

Theorem 3 *Let E be an elliptic curve over $\mathbf{Q}$, not of CM-type of square-free conductor, parametrized by the cuspidal newform f_E of weight two and level N (with integral Fourier coefficients). Let p be a prime of good, ordinary, reduction for E,; let $\mathbf{T}(N;p^\infty) = \mathbf{T}(N;p^\infty;\mathbf{Z}_p)$ be the Λ-algebra of Hecke operators in*

$$\lim_r H_1(X(\Gamma_0(N)\cap\Gamma_1(p^r));\mathbf{Z}_p)$$

and $m = m_{E,p} \subset \mathbf{T}(N;p^\infty)$ the maximal ideal corresponding to f_E mod p (I.e., $m_{E,p}$ is the ideal generated by p, $T_\ell - a_\ell$ for prime numbers ℓ not dividing pN, and $U_q - a_q$ for prime numbers q dividing pN, where the a_ℓ's and a_q's are the Fourier coefficients of f_E.)

Then for a collection of prime numbers p of Dirichlet density one, we have that the Λ-algebra $\mathbf{T}(N;p^\infty)^{\mathrm{ord}}_M$ is of rank one, where M is the maximal ideal corresponding to the ordinary eigenform on $\Gamma_0\cap\Gamma_1(p)$ obtained by lifting f_E. In particular, we have:

$$\Lambda = \mathbf{T}(N;p^\infty)^{\mathrm{ord}}_M.$$

Proof: Let p be a prime number satisfying the following conditions:

1. $p \geq 7$,
2. p is of good, ordinary, reduction for E,
3. p does not divide the discriminant of the Hecke algebra $\mathbf{T}(N);\mathbf{Z})$ (generated over $\mathbf{Z}$ by the usual suspects) acting faithfully on $H_1(X_0(N);\mathbf{Z}))$
4. $a_p \neq \pm 1$.

Note that if $p \geq 7$ and $a_p \neq \pm 1$, the "Riemann Hypothesis" for elliptic curves over finite fields guarantees that $a_p \not\equiv \pm 1 \bmod p$, a fact that will be relevant below.

Let a *dash*, $'$, denote sub-algebras of Hecke algebras, but deprived of either T_p if p does not divide the level and or U_p if p does divide the level. So we have

$$\mathbf{T}'(N;p);\mathbf{Z}_p) \subset \mathbf{T}(N;p);\mathbf{Z}_p))$$

and

$$\mathbf{T}'(N;\mathbf{Z}_p) \subset \mathbf{T}(N);\mathbf{Z}_p)).$$

We have a natural surjection

$$\mathbf{T}'(N;p);\mathbf{Z}_p)) \to \mathbf{T}'(N;\mathbf{Z}_p)),$$

and a natural injection

$$\mathbf{T}'(N;p);\mathbf{Z}_p)) \to \mathbf{T}(N;\mathbf{Z}_p)).$$

Lemma 1. Let $m = m_{E,p}$, and m' following the convention above. The natural injection above induces isomorphisms

$$\mathbf{T}'(N;\mathbf{Z}_p))_{m'} \cong \mathbf{T}(N;\mathbf{Z}_p)_m \cong \mathbf{Z}_p.$$

This comes from the fact that $m = m_{E,p}$ is "new" in level N, and that any newform in a given level is determined (among newforms of its level) by all but a finite number of its Hecke eigenvalues, and finally that p doesn't divide the discriminant of $\mathbf{T}(N;\mathbf{Z})$.

We now want to "raise level."

The basic fact here is that working in weight k on $\Gamma_0(Np)$ we have that (if w_p is the Atkin-lehner involution)

$$U_p + w_p p^{k-2/2} : S_k(\Gamma_0(Np)) \to S_k(\Gamma_0(N))$$

and therefore any p-new eigenform of weight two on $\Gamma_0(Np)$ has U_p-eigenvalue ± 1, or we might also say that for any $\alpha \neq \pm 1$ we have that

$$H_1(X_0(Np);\bar{\mathbf{Z}}_p)[U_p - \alpha] \subset H_1(X_0(Np);\bar{\mathbf{Z}}_p)$$

lies in the subspace of p-old forms.

As a result, any p-ordinary newform of weight two on $\Gamma_0(N)$ such that $a_p \not\equiv \pm 1 \bmod p$ has the property that when it is lifted to an ordinary eigenform on $\Gamma_0(Np)$ it will not be congruent mod p to any newform of weight two on $\Gamma_0(Np)$. The upshot of this is, if we let $M \subset \mathbf{T}(N;p;\mathbf{Z}_p))$ denote the maximal ideal generated by $m' \subset \mathbf{T}'(N;p;\mathbf{Z}_p)) \subset \mathbf{T}(N;p;\mathbf{Z}_p))$ and by $U_p - \alpha_p$ where α_p is the p-unit root of Frobenius action on the elliptic curve E, then

Lemma 2. The natural surjection above induces isomorphisms

$$\mathbf{T}(N;p;\mathbf{Z}_p))^{\rm ord}_M \cong \mathbf{T}'(N;p;\mathbf{Z}_p))^{\rm ord}_{m'} \cong \mathbf{T}'(N;\mathbf{Z}_p)^{\rm ord}_{m'} \cong \mathbf{T}(N;\mathbf{Z}_p)^{\rm ord}_m \cong \mathbf{Z}_p.$$

What remains to see is that

$$\mathbf{T}(N;p;\mathbf{Z}_p))^{\rm ord}_M \cong \mathbf{Z}_p$$

is really enough to guarantee that $r_p(M) = 1$ (which comes from the application of Nakayama's lemma that I did in class) and that the conditions enumerated in the theorem correspond to a collection of primes of density one.

2 The Eisenstein and cuspidal loci

Consider $Y_1(Np^r) \subset X_1(Np^r) = Y_1(Np^r) \sqcup \mathrm{Cusp}_1(Np^r)$ and the corresponding exact sequence of homology

$$0 \to H_2(X_1(p^r);\mathbf{Z}_p) \to H_2(X_1(p^r), Y_1(Np^r);\mathbf{Z}_p) \to H_1(Y_1(Np^r);\mathbf{Z}_p) \to H_1(X_1(Np^r);\mathbf{Z}_p) \to 0,$$

which, more explicitly, gives us

$$0 \to \mathbf{Z}_p \to \mathbf{Z}_p[\mathrm{Cusp}_1(Np^r)] \to H_1(Y_1(Np^r); \mathbf{Z}_p) \to H_1(X_1(Np^r); \mathbf{Z}_p) \to 0.$$

Discuss fusion between the cuspidal part of the Hecke algebra and the Eisenstein part. The quotients

$$\mathbf{T}^{\mathrm{ord}} \to \mathbf{T}^{\mathrm{ord,Cusp}}$$

and

$$\mathbf{T}^{\mathrm{ord}} \to \mathbf{T}^{\mathrm{ord,Eis}}.$$

3 The Galois action

Our next view of $H_1(X_1(Np^r), \mathbf{Z}_p)$ is as the projective limit of the p-power torsion in the jacobian of $X_1(Np^r)$. So let $J_1(Np^r) =$: the jacobian of $X_1(Np^r)$, viewed as abelian variety over $\mathbf{Q}$ and let $J_1(Np^r)[p^\nu]$ be the kernel of multiplication by p^ν in the group, $J_1(Np^r)(\bar{\mathbf{Q}})$, of $\bar{\mathbf{Q}}$-rational points of $J_1(Np^r)$. By standard comparison theorems, if we view $\bar{\mathbf{Q}}$ as a subfield of $\mathbf{C}$—which is our convention, in any case—we may identify

$$H_1(X_1(Np^r), \mathbf{Z}_p) \cong \lim_{\nu} J_1(Np^r)[p^\nu]$$

from which $H_1(X_1(Np^r), \mathbf{Z}_p)$ inherits a continuous $G_{\mathbf{Q},S}$-action where S is the set of primes dividing $p \cdot N \cdot \infty$. Passing to the ordinary quotient $H_1(X_1(Np^r), \mathbf{Z}_p)^{\mathrm{ord}}$ and noting that the Galois action commutes with the action of the Hecke operators, we have, for example when $N = 1$ and m is non-Eisenstein,

$$\rho_m^{\mathrm{ord}} : G_{\mathbf{Q},\{p,\infty\}} \longrightarrow \mathrm{Aut}_{\mathbf{T}_m^{\mathrm{ord}}}(\lim_{\nu} J_1(Np^r)[p^\nu])_m^{\mathrm{ord}}) \quad \cong \quad \mathbf{GL}_2(\mathbf{T}_m^{\mathrm{ord}}).$$

For this section continue to let $N = 1$.

1. **The determinant.**

$$\det(\rho_m^{\mathrm{ord}}) : G_{\mathbf{Q},\{p,\infty\}} \longrightarrow (\mathbf{T}_m^{\mathrm{ord}})^*$$

   [can be made explicit, as a specific finite character times the wild p-adic cyclotomic character, $\det(\rho_m^{\mathrm{ord}}) : G_{\mathbf{Q},\{p,\infty\}} \to \Lambda^*$].

2. **The action of the decomposition group at each prime of Q under the representation ρ_m^{ord}:**

- **The prime at infinity**
  Complex conjugation does not act as a scalar, and if

$$(H_1(X_1(p^r), \mathbf{Z}_p)_m^{\mathrm{ord}})^{\pm}$$

  are the $\pm$-eigenspaces for complex conjugation, then each is free of rank one over $\mathbf{T}_m^{\mathrm{ord}}$.
- **The prime** p
  The image of "the" decomposition group D_p at p under ρ_m^{ord} is contained in a Borel subgroup

$$g \mapsto \begin{pmatrix} \alpha(g) & \beta(g) \\ 0 & \delta(g) \end{pmatrix}$$

  in $\mathbf{GL}_2(\mathbf{T}_m^{\mathrm{ord}})$ for $g \in D_p$, and the character

$$\delta : D_p \to (\mathbf{T}_m^{\mathrm{ord}})^*$$

  is unramified. If $Frob_p \in D_p$ is a choice of Frobenius element at p, then

$$\delta(Frob_p) \in (\mathbf{T}_m^{\mathrm{ord}})^*$$

  is independent of the choice of p and, moreover, we have:

$$\delta(Frob_p) = U_p \ \in \ (\mathbf{T}_m^{\mathrm{ord}})^*.$$

- **A prime** $\ell \neq p, \infty$
  Here ρ_m^{ord} is unramified at ℓ and if $Frob_\ell$ is a choice of Frobenius at ℓ, we have the formula

$$\mathrm{Trace}_{\mathbf{T}_m^{\mathrm{ord}}}(\rho_m^{\mathrm{ord}}(Frob_\ell)) \ = \ T_\ell \ \in \ \mathbf{T}_m^{\mathrm{ord}}.$$

4 The Eisenstein ideal

Let$N = 1$ and let us cut $\mathbf{T}^{\mathrm{ord}}$ in terms of tame characters,

$$\mathbf{T}^{\mathrm{ord}} = \prod_{i \in \mathbf{Z}/(p-1)\mathbf{Z}} \mathbf{T}_{\{i\}}^{\mathrm{ord}}.$$

Here $\mathbf{T}_{\{i\}}^{\mathrm{ord}}$ is the component where the tame diamond operators act through the character ω^i.

Define the **Eisenstein ideal** $I_{\{i\}} \subset \mathbf{T}_{\{i\}}^{\mathrm{ord}}$ to be the ideal generated by the elements

$$T_\ell - (1 + \langle \ell \rangle \ell^{-1})$$

for primes $\ell \neq p$ and by $U_p - 1$.

We have

$$0 \to I_{\{i\}} \to \mathbf{T}^{\mathrm{ord}}_{\{i\}} \to \mathbf{T}^{\mathrm{ord,Eis}}_{\{i\}} \to 0$$

(with evident notation).

The real surprise is

Theorem 4 *Let $i \in 2\mathbf{Z}/(p-1)\mathbf{Z}$ and suppose that $i \neq 0$. If $I^{\mathrm{Cusp}}_{\{i\}}$ denotes the image of $I_{\{i\}}$ in the quotient Λ-algebra $\mathbf{T}^{\mathrm{ord,Cusp}}_{\{i\}}$ (again, evident notation) then,*

$$\frac{\Lambda}{L_{p,\{i\}}\Lambda} \cong \frac{\mathbf{T}^{\mathrm{ord,Cusp}}_{\{i\}}}{I^{\mathrm{Cusp}}_{\{i\}}}.$$

Discuss!

5 Universal deformations

Here I assume that the general issues about Galois deformations are understood, and that it is known that

- if k is a finite field of characteristic p, and S is a finite set of primes including p and ∞, and

$$\bar{\rho} : G_{\mathbf{Q},S} \to \mathrm{GL}_2(k)$$

  is an odd continuous representation such that the centralizer of its image consists only in scalars (e.g., this will be the case if $\bar{\rho}$ is absolutely irreducible) then there is a *universal Galois deformation*

$$\rho_{\mathrm{univ}} : G_{\mathbf{Q},S} \to \mathrm{GL}_2(R(\bar{\rho}))$$

  where $R(\bar{\rho})$ is a complete noetherian local ring with residue field k, and is $R(\bar{\rho})/pR(\bar{\rho})$ of Krull dimension ≥ 3; and

- if the deformation problem is *unobstructed* [explain this a bit] then

$$R(\bar{\rho}) \cong W(k)[[T_1, T_2, T_3]],$$

  i.e., $R(\bar{\rho})$ is a power series ring in three variables over the ring of Witt vectors of k; and

- if $\bar{\rho}$ is (strictly) ordinary, in the sense that the image of "a" decomposition group at p is contained in a Borel subgroup conjugate to the group of upper triangular matrices, and if the quotient-character on the decomposition group is unramified, then it makes sense to define the *universal ordinary deformation ring* of $\bar{\rho}$ which is the maximal quotient ring

$$R(\bar{\rho}) \longrightarrow R(\bar{\rho})^o$$

such that the representation

$$\rho^o_{\text{univ}} : G_{\mathbf{Q},S} \to \mathrm{GL}_2(R(\bar\rho)^o)$$

induced from ρ_{univ} is strictly ordinary; and

- if $\bar\rho$ is "modular", and is therefore the residual representation attached to a maximal ideal $m \subset \mathbf{T}^{\text{ord}}$ as above, then we have a surjection of Λ-algebras,

$$R(\bar\rho)^o \longrightarrow \mathbf{T}^{\text{ord}}_m.$$

Ignorable Side comment on residual irreducibility: *Things work well, in the above theorem, if m is a maximal ideal attached to an absolutely irreducible residual representation. If not, it pays to work with universal deformations of pseudo-representations (cf. Chapter 5 of [6]) and to pass to the rigid-analytic universal deformation space. Here things behave very nicely, and to cite a recent result where pseudo-representations are key, see [1] where it is proved that the eigencurve is smooth at the evil Eisenstein points and where the authors give necessary and sufficient conditions for étaleness of the map to the weight space at these points in terms of p-adic zeta values.*

6 An $R(\bar\rho)^o = \mathbf{T}^{\text{ord}}_m$ theorem

Theorem 5 *Let $S = \{p, \infty\}$ and suppose that $\bar\rho$ is "modular, " ordinary, and satisfied the condition that if $L = \mathbf{Q}(\sqrt{p^*})$ with $p* := (-1)^{\frac{p-1}{2}}p$ the restriction of $\bar\rho$ to $G_L := \mathrm{Gal}(\bar Q/L)$ is absolutely irreducible. Then the natural homomorphism of Λ-algebras,*

$$R(\bar\rho)^o \longrightarrow \mathbf{T}^{\text{ord}}_m$$

is an isomorphism.

A proof of this theorem is given in [30], the point being that if $a_2 \subset \Lambda$ is the kernel of specialization to weight two, by the modularity theorem we have that the surjective homomorphism above induces an isomorphism,

$$R(\bar\rho)^o/a_2R(\bar\rho)^o \cong \mathbf{T}^{\text{ord}}_m/a_2\mathbf{T}^{\text{ord}}_m,$$

and so an application of Nakayama's lemma (using finite flatness of the Λ-module $\mathbf{T}^{\text{ord}}_m$) gives our theorem. [Explicitly, viewing $\mathbf{T}^{\text{ord}}_m$ as a free Λ-module of finite rank, and $R(\bar\rho)^o$ as a Λ-module of finite type, by the isomorphism displayed just above, we have a Λ-module homomorphism

$$\mathbf{T}^{\text{ord}}_m \to R(\bar\rho)^o/a_2R(\bar\rho)^o$$

which may be lifted (since $\mathbf{T}^{\text{ord}}_m$ is a free Λ-module) to a Λ-module homomorphism

$$\mathbf{T}^{\text{ord}}_m \to R(\bar\rho)^o$$

which is surjective by straightforward application of Nakayama's lemma, and then seen to be injective as well by considering the composition

$$\mathbf{T}_m^{\text{ord}} \to R(\bar{\rho})^o \to \mathbf{T}_m^{\text{ord}}.]$$

7 The structure of the universal deformation ring

When is the universal deformation problem unobstructed? Equivalently, for which residual representations $\bar{\rho}$ do we have that $R(\bar{\rho})$ is isomorphic to a power series ring in three variables over the ring of Witt vectors of the residue field?

Theorem 6 *Let $S = \{p, \infty\}$, and*

$$\bar{\rho} : G_{\{}\mathbf{Q}, S\} \to \mathrm{GL}_2(\mathbf{F}_p)$$

a continuous representation that is modular, ordinary, absolutely irreducible when restricted to G_L where $L = \mathbf{Q}(\sqrt{p^*})$ and of determinant equal to ω^i where i is not congruent to $0, \pm 1$, or $\frac{p-1}{2}$ modulo $p-1$. Then the universal deformation ring attached to $\bar{\rho}$ is unobstructed, and

$$R(\bar{\rho}) \cong \Lambda[[T_1, T_2]],$$

i.e., $R(\bar{\rho})$ is isomorphic, as Λ-algebra to a power series ring in two variables over Λ.

The proof is given by the main proposition of section 32 of [19].

8 The ordinary part of the eigencurve

To focus on the structure, let us assume that $N = 1$ but everything that we say has an easy modification to cover more general tame levels. We have the semi-local ring

$$\mathbf{T}^{\text{ord}} := \prod_m \mathbf{T}_m^{\text{ord}}$$

where the product is over the finitely many maximal ideals $m \subset \mathbf{T}^{\text{ord}}$.

We view $\mathbf{T}^{\text{ord}}$ as a finite flat $\mathbf{Z}_p[[\mathbf{Z}_p^*]] = \mathbf{Z}_p[\mathbf{F}_p^*] \otimes_{\mathbf{Z}_p} \Lambda$-algebra, where $\mathbf{Z}_p^*$ acts as diamond operators (writing $\mathbf{Z}_p^* = \mathbf{F}_p^* \times \Gamma$ the group $\mathbf{F}_p^*$ constitutes the tame diamond operators, and Γ the wild ones).

We therefore can cut $\mathbf{T}^{\text{ord}}$ up slightly differently; i.e.,

$$\mathbf{T}^{\text{ord}} := \prod_{i \in \mathbf{Z}/(p-1)\mathbf{Z}} \mathbf{T}^{\text{ord},\{\text{i}\}}$$

where

$$\mathbf{T}^{\mathrm{ord},\{\mathrm{i}\}} := \mathbf{T}^{\mathrm{ord}} \otimes_{\mathbf{Z}_p[[\mathbf{Z}_p^*]]} \Lambda,$$

and where the mapping

$$\mathbf{Z}_p[[\mathbf{Z}_p^*]] \to \Lambda$$

is induced by the i-th power of the Teichmüller character. If m has the property that its corresponding residual representation has determinant ω^{1-i} then $\mathbf{T}_m^{\mathrm{ord}}$ is a factor of $\mathbf{T}^{\mathrm{ord},\{\mathrm{i}\}}$; in this case, for short, let us say that "m belongs to $\{i\}$."

We form the corresponding spectrum and get the (cuspidal part of the) **ordinary eigencurve**,

$$X^{\mathrm{ord}} := \mathrm{Spec}(\mathbf{T}^{\mathrm{ord}}).$$

We have the corresponding decompositions,

$$X^{\mathrm{ord}} = \bigsqcup_{i \in \mathbf{Z}/(p-1)\mathbf{Z}} X^{\mathrm{ord},\{\mathrm{i}\}},$$

and

$$X^{\mathrm{ord},\{\mathrm{i}\}} = \bigsqcup_{m \text{ belongsto } \{i\}} X_m^{\mathrm{ord}},$$

the X_m^{ord} being irreducible, and finite flat over weight space $W := \mathrm{Spec}(\Lambda)$. For each weight $w \in W$ our curve "parametrizes" a finite number of eigenforms (in fact, the "same" number as one moves from weight to weight in W). Discuss its relationship to the full universal deformation space.

We can cut X^{ord} up slightly differently:

$$X^{\mathrm{ord}} = X^{\mathrm{ord}}_{\text{res.irr.}} \sqcup X^{\mathrm{ord}}_{\text{res.red.}}$$

where $X^{\mathrm{ord}}_{\text{res.irr.}}$ collects the irreducible components X_m^{ord} such that the residual representation attached to m is absolutely irreducible, and $X^{\mathrm{ord}}_{\text{res.red.}}$ collects the others.

Over $X^{\mathrm{ord}}_{\text{res.irr.}}$ we have a (trivial) vector bundle of rank two, with linear $G_{\mathbf{Q},\{p,\infty\}}$-action. Over $X^{\mathrm{ord}}_{\text{res.red.}}$ we have to make do with slightly less, i.e., a pseudo-representation of rank two (is it realizable—if not canonically–by a vector bundle of rank two over the rigidn analytic space with linear $G_{\mathbf{Q},\{p,\infty\}}$-action?

We want to understand the "curve" X^{ord} as a parameter space. Specifically, we wish to see (recalling that we are restricting, for the moment our attention to tame level one)

1. all p-adic modular (ordinary, overconvergent) eigenforms parametrized by X^{ord},
2. the p-adic modular (ordinary, overconvergent) eigenforms of half-integral weight, corresponding to the Shimura lift of classical ordinary eigenforms parametrized by X^{ord} (work of Jochnowitz, Pancuskin, Ramsey),
3. the symmetric squares of p-adic modular (ordinary, overconvergent) eigenforms parametrized by X^{ord},

4. the p-adic analytic L-function associated to an ordinary p-adic eigenform relative to the standard representation, parametrized by X^{ord} (see, for example, the unpublished note *Anomalous eigenforms and the two-variable p-adic L-function* on the web-page of this course).

5. the p-adic analytic L-function associated to an ordinary p-adic eigenform relative to the symmetric square representation, parametrized by X^{ord} (see recent work of Walter Kim)

6. p-adic skew-Hermitian organization modules, as vector bundles of rank two on X^{ord} yielding Selmer modules[2] parametrized by X^{ord},

7. the p-adic arithmetic L-function relative to the standard, and other, GL_2- representations, associated to an ordinary p-adic eigenform parametrized by X^{ord},

8. the corresponding Langlands (but p-adic Banach) automorphic representations of GL_2, again parametrized by X^{ord} (current work of Breuil, Berger, Colmez, Emerton, Kisin, Schneider, Teitelbaum, ...).

Comment on

- seeing X^{ord} is a piece of the (finite slope) eigencurve,
- the general program of consructing p-adic analytic L-functions associated to an ordinary p-adic eigenform relative to *every* representation of $GL(2)$, all of these L-functions parametrized by the eigencurve,
- and on seeing the Langlands program formulated, more generally, in terms of eigenvarieties.

[2] current work of Pottharst which gives such a structure—for $N = 1$ and over components X_m^{ord} which are irreducible and such that the residual representation attached to m is absolutely irreducible, and for any quadratic imaginary field K where p splits.

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