Introductory comments on the eigencurve

Handout # 5:

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(These are brief indications, hardly more than an annotated list, of topics mentioned in my lectures.)

1 The basic Hecke diagram

As before we fix $N = 1$, $p \geq 5$ and $i \not\equiv 0 \bmod p-1$. But let $i \equiv 2 \bmod p-1$.

Recall T_0, the 0-*th Hecke operator*. We have that the cuspidal quotient of the Hecke algebra is obtained by dividing by the principal ideal generated by T_0, giving the exact sequence:

$$0 \to (T_0) \to \mathbf{T}^{\mathrm{ord}}_{\{i\}} \to \mathbf{T}^{\mathrm{ord}}_{\mathrm{cusp},\{i\}} \to 0.$$

Define the *Eisenstein ideal* $I_{\{i\}}$ in $\mathbf{T}^{\mathrm{ord}}_{\{i\}}$ to be the ideal generated by $\eta_\ell \in \mathbf{T}^{\mathrm{ord}}_{\{i\}}$ for all primes ℓ where

$$\eta_\ell := T_\ell - (1 + \ell\langle\ell\rangle)$$

for ℓ different from p and $\eta_p = U_p - 1$.

Define $\mathbf{T}_{\mathrm{Eis}}$ as the quotient ring:

$$0 \to I_{\{i\}} \to \mathbf{T}^{\mathrm{ord}}_{\{i\}} \to \mathbf{T}_{\mathrm{Eis},\{i\}} \to 0.$$

Theorem 1 *1. The natural ring-homomorphism*

$$h : \Lambda_{\{i\}} \to \mathbf{T}_{\mathrm{Eis}\{i\}}$$

is an isomorphism.

2. The image under h of the 0-th Hecke operator T_0 in

$$\Lambda_{\{i\}} = \mathbf{T}_{\mathrm{Eis},\{i\}} = \mathbf{T}^{\mathrm{ord}}_{\{i\}}/I_{\{i\}}$$

is $\frac{1}{2}L_{p,\{i\}} \in \Lambda_{\{i\}}$ where $L_{p,\{i\}}$ is the Leopoldt-Kubota function in $\Lambda_{\{i\}}$.

[*Recall the proof:*

1. The natural homomorphism $h : \Lambda_{\{i\}} \to \mathbf{T}^{\mathrm{ord}}_{\{i\}}/I_{\{i\}}$ is surjective, and we must show that the kernel of h is 0; this we do by recalling that for every $k \geq 2$ and $k \equiv i \bmod p-1$ the Eisenstein series $mathcalE_k$ lies in $\mathcal{V}_i$, and is in the kernel of $I_{\{i\}}$ (under the action of $\mathbf{T}^{\mathrm{ord}}_{\{i\}}$). It follows that $\ker(h)$ is in the kernel of the specialization mapping of $\Lambda_{\{i\}}$ to weight k for all $k \geq 2$ and $k \equiv i \bmod p-1$. Hence $\ker(h) = 0$.
2. This follows from the q-expansion principle, and the fact that $\frac{1}{2}L_{p,\{i\}} \in \Lambda_{\{i\}}$ is the constant term of the "Λ-adic" Eisenstein series $\mathcal{E}_{\{i\}}$.]

Express this in terms of Specs and rigid analytic spaces X_*. So:

$$W_{\{i\}} := \mathrm{Spec}(\Lambda_{\{i\}}),$$

$$X^{\mathrm{ord}}_{\{i\}} := \mathrm{Spec}(\mathbf{T}^{\mathrm{ord}}_{\{i\}}),$$

$$X^{\mathrm{ord}}_{\mathrm{cusp},\{i\}} := \mathrm{Spec}(\mathbf{T}^{\mathrm{ord}}_{\mathrm{cusp},\{i\}}),$$

$$X_{\mathrm{Eis},\{i\}} := \mathrm{Spec}(\mathbf{T}_{\mathrm{Eis},\{i\}}),$$

Suppress the subscripts $\{i\}$. Define *Leopoldt-Kubota point* in

$$X^{\mathrm{ord}}_{\mathrm{cusp},\{i\}} \cap X_{\mathrm{Eis},\{i\}} \quad \subset \quad X^{\mathrm{ord}}_{\{i\}}.$$

Summary

- $T_0 = 0$ cuts out $X^{\mathrm{ord}}_{\mathrm{cusp}}$.
- Discuss $\eta_\ell = 0$ in $X^{\mathrm{ord}}_{\mathrm{cusp}}$ and its relationship to the Leopoldt-Kubota points.

- Define:

$$D(\ell) := \mathrm{Norm}_{\{\mathbf{T}_{\mathrm{cusp}}/\Lambda\}}(\eta_\ell) \in \Lambda.$$

Question: Is the ideal generated by $L_{p,\{i\}}$ in Λ equal to the ideal generated by all the $D(\ell)$s?

Corollary: $L_{p,\{i\}}$ divides $D(\ell)$ in Λ.

2 Anomalous eigenforms

Specifically, $D(p) = \mathrm{Norm}_{\mathbf{T}_{\mathrm{cusp}}/\Lambda}(U_p - 1)$.

Define *anomalous eigenform* (i.e., one fixed by U_p).

Discuss *classically irregular* and *theta-irregular*, and, for $p < 200$ there are eight *sporadically irregular* zeroes.

3 Igusa tower

4 Associated representations

Theorem. There is a vector bundle of rank two over the normalization, $\tilde{X}^{\mathrm{ord}}_{\mathrm{cusp}}$ of $X^{\mathrm{ord}}_{\mathrm{cusp}}$ with an $\mathcal{O}_{\tilde{X}^{\mathrm{ord}}_{\mathrm{cusp}}}$-Linear action of $G_{\mathbf{Q},\{p,\infty\}}$ satisfying the three properties previously described (regarding *determinant, Eichler-Shimura relations, and ordinary decomposition group behavior*) and pinned down already by the Eichler-Shimura relations.

Discuss when reducible!

5 Connection with classical Iwasawsa theory.

6 The eigencurve

We see X^{ord} as a piece of the (finite slope) eigencurve X. We want to understand the curve X as a parameter space. Specifically, we wish to see (recalling that we are restricting, for the moment our attention to tame level one)

1. all p-adic modular (finite slope, overconvergent) eigenforms parametrized by X,

2. the p-adic modular (finite slope, overconvergent) eigenforms of half-integral weight, corresponding to the Shimura lift of classical finite slope eigenforms parametrized by X^{ord} (work of Jochnowitz, Pancuskin, Ramsey),

3. the symmetric squares of p-adic modular (finite slope, overconvergent) eigenforms parametrized by X^{ord},

4. the p-adic analytic L-function associated to an finite slope p-adic eigenform relative to the standard representation, parametrized by X (see, for example, the unpublished note *Anomalous eigenforms and the two-variable p-adic L-function parametrized by* X^{ord} on the web-page of this course).

5. the p-adic analytic L-function associated to an finite slope p-adic eigenform relative to the symmetric square representation, parametrized by X (see recent work of Walter Kim for X^{ord})

6. p-adic skew-Hermitian organization modules, as vector bundles of rank two on X yielding Selmer modules (see current work of Pottharst for X^{ord}),

7. the p-adic arithmetic L-function relative to the standard, and other, GL_2- representations, associated to an finite slope p-adic eigenform parametrized by X.

Comment on

- the general program of constructing p-adic analytic L-functions associated to an finite slope p-adic eigenform relative to *every* representation of $GL(2)$, all of these L-functions parametrized by the eigencurve,

- and on seeing the Langlands program formulated, more generally, in terms of eigenvarieties.

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