

Math 254z Spring, 2006

Topics in p -adic automorphic forms

February 3, 2006

Some very rough introductory material

For a full description of the specific topics consult the web-site

<http://www.math.harvard.edu/ev/>

and the various links on that page. This seminar-course will include the following lecture series:

- Jacques Tilouine (p -adic Siegel modular forms)
- Gaetan Chenevier (The eigenvarieties of definite unitary groups)
- Peter Schneider (p -adic Banach representations)
- Mark Kisin (The eigencurve via Galois representations)
- Laurent Berger (Aspects of the p -adic Langlands program)
- Christophe Breuil (Aspects of the p -adic Langlands correspondence)
- Shekar Khare (Serre's Conjecture)
- Michael Harris (On the stable trace formula for unitary groups with applications to construction of Galois representations)
- Matthew Emerton (Local-global compatibility in the p -adic Langlands correspondence for GL_2 over \mathbf{Q})
- Haruzo Hida (The \mathcal{L} -invariant and Galois deformation theory)
- Jeremy Teitelbaum (Title: TBA)

1 Introduction

As a general introduction to the subject of the course we will formulate three theorems (without proofs).

By $G_{\mathbf{Q}}$ let us mean the Galois group of $\bar{\mathbf{Q}}$ over \mathbf{Q} where $\bar{\mathbf{Q}} \subset \mathbf{C}$ is the algebraic closure of \mathbf{Q} in \mathbf{C} . If S is a finite set of primes $G_{\mathbf{Q},S}$ denotes the quotient of $G_{\mathbf{Q}}$ that is the Galois group of the maximal extension of \mathbf{Q} in $\bar{\mathbf{Q}}$ that is unramified at all primes outside S . Equivalently, it is the quotient of $G_{\mathbf{Q}}$ by the closed normal subgroup generated by the inertia subgroups at primes of residual characteristic not in S . The inertia subgroup at the *infinite prime* is the group of order two generated by complex conjugation.

By way of introduction to our course I want to consider a certain Galois extension K/\mathbf{Q} of degree 13,200. It provides an example where a good many results and techniques come together. Specifically, put $S = \{11, \infty\}$; I wish to find a continuous homomorphism

$$\bar{\rho} : G_{\mathbf{Q},S} \rightarrow \mathrm{GL}_2(\mathbf{F}_{11})$$

with these properties:

1. The homomorphism $\bar{\rho}$ is surjective,
2. The determinant homomorphism,

$$\det(\bar{\rho}) : G_{\mathbf{Q},S} \rightarrow \mathrm{GL}_1(\mathbf{F}_{11}) = \mathbf{F}_{11}^*,$$

is the cyclotomic character $\bar{\chi}_{11}$. That is, for $g \in G_{\mathbf{Q}}$ and an 11-th root of unity $\zeta_{11} \in \bar{\mathbf{Q}}$ we have

$$g(\zeta_{11}) = \zeta_{11}^{\bar{\chi}_{11}(g)},$$

3. The restriction of $\bar{\rho}$ to a decomposition group at 11 has image contained in a subgroup of the form

$$\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$$

that cuts out a totally ramified extension of \mathbf{Q}_{11} of degree $110 = 10 \times 11$.

Reserve the letter \bar{V} for the $G_{\mathbf{Q},S}$ -representation space corresponding to $\bar{\rho}$. That is, \bar{V} is a two-dimensional \mathbf{F}_{11} -vector space, $\bar{V} = \mathbf{F}_{11} \times \mathbf{F}_{11}$, with \mathbf{F}_{11} -linear action by $G_{\mathbf{Q},S}$ given by $\bar{\rho}$.

Our field extension K/\mathbf{Q} is the splitting field of the representation $\bar{\rho}$.

Theorem 1 *There is a unique $G_{\mathbf{Q}}$ -representation $\bar{\rho}$ (up to conjugation) that satisfies conditions (1), (2), (3).*

Consequently, of course, the field extension K/\mathbf{Q} is uniquely characterized by its role as splitting field for a representation satisfying those conditions. This striking theorem follows from recent work of Khare on Serre's conjecture [*See Khare's web-page!*] and therefore it requires, as well, many of the ingredients of Khare's work. But even if you knew this theorem, how would you go about studying this impressively large K/\mathbf{Q} ?

2 Traces.

Let $\bar{\rho}$ be the representation discussed in the previous section. For ℓ a prime number different from 11 put

$$a_{\ell} := \text{Trace}(\bar{\rho}(\text{Frob}_{\ell})) \in \mathbf{F}_{11}.$$

Extend this ($\ell \mapsto a_{\ell}$) to a mapping,

$$n \mapsto a_n \in \mathbf{F}_{11},$$

for all non-negative integers n by setting

$$a_{11^{\nu}} = 1$$

for all $\nu \geq 0$,

$$a_{\ell^{\nu}} := a_{\ell^{\nu-1}}a_{\ell} - \ell a_{\ell^{\nu-2}}$$

for all prime numbers $\ell \neq 11$, and finally

$$a_{n \cdot m} := a_n \cdot a_m$$

if n and m are relatively prime.

These are the recursive relations that would transpire if you simply formally expanded the infinite product

$$\prod_{p \neq 11} \frac{1}{1 - a_p p^{-s} + p^{1-2s}} \cdot \frac{1}{1 - 11^{-s}}$$

as an infinite sum

$$\sum_{n \geq 0} a_n n^{-s}.$$

Table 1: Ramanujan's τ -function modulo 11

n:	1	2	3	4	5	6	7	8	9	10	11	12	13
a_n :	1	-2	-1	2	1	2	-2	0	4	-2	1	-2	4

Go one step further and form the utterly formal *inverse Mellin transform* to obtain the power series

$$\sum_{n \geq 0} a_n q^n \in \mathbf{F}_{11}[[q]]$$

knowledge of which would, by the classical Brauer-Nesbitt Theorem, completely pinpoint the Galois representation, and therefore our extension K/\mathbf{Q} . Having said this, it is time to give you the wonderful formula for all these a_n 's, namely

$$\sum_{n \geq 0} a_n q^n = q \prod (1 - q^n)^{24} \equiv q \prod (1 - q^n)^2 \cdot \prod (1 - q^{11n})^2 \in \mathbf{F}_{11}[[q]].$$

The first equality reminds us that a_n is given by τ_n taken modulo 11, where $n \mapsto \tau_n$ is Ramanujan's τ -function.

3 Construction via elliptic curves.

Can we find an elliptic curve E over \mathbf{Q} such that the natural $G_{\mathbf{Q}}$ -representation on $E[11]$, the kernel of multiplication by 11 in E , is equivalent to the irreducible $G_{\mathbf{Q}}$ -representation \bar{V} ? This, properly asked, is a moduli question answered by the \mathbf{Q} -rational points of a *modular curve* $X(\bar{V})$, defined over \mathbf{Q} whose base change to \mathbf{C} is equal to the classical $X(11)$ (and hence is of genus 26). There are three known rational points on this curve of genus 26, given by

$$X_1(11) \rightarrow X_0(11) \rightarrow X_{-1}(11)$$

the morphisms being isogenies of degree 5. For each of these three elliptic curves E we have an isomorphism of $G_{\mathbf{Q}}$ -representation spaces

$$\bar{V} \cong E[11]$$

and in particular, we may view our field extension K/\mathbf{Q} as generated by the “coordinates” of the points of order 11 in E , where E is any of the three elliptic curves listed; e.g., $E = X_1(11)$ whose equation is given by

$$y^2 + y = x^3 + x.$$

Are these three all the \mathbf{Q} -rational points of $X(\bar{V})$? This, by the way, is a particularly incestuous example, since the rational points are all isogenous to $X_0(11)$ and the jacobian of $X(\bar{V})$ is isogenous to 26 copies of $X_0(11)$.

4 Modular forms modulo p

The power series

$$\bar{f} := \sum_{n \geq 0} a_n q^n \in \mathbf{F}_{11}[[q]]$$

is a perfect example of (the Fourier series of) a *modular form* modulo $p = 11$. (More specifically, it is a modular form mod p on $\Gamma_0(p)$ that is a normalized cuspidal eigenform for all the Hecke operators T_ℓ with $\ell \neq p$ and for the Atkin-Lehner operator U_p .) This notion of “modular form mod p ” can be taken in any of the various senses, the most elementary being *modular form modulo p in the sense of Serre*, i.e., the reduction modulo a maximal ideal $m \subset \mathcal{O}$ of residual characteristic p of the Fourier series of a classical modular form with Fourier coefficients in the ring $\mathcal{O} \subset \mathbf{C}$.

Serre, in fact, has computed the ring of modular forms modulo $p = 11$ of level one. This ring, viewed as graded \mathbf{F}_{11} -algebra with gradation modulo 10, is $\mathbf{F}_{11}[Q, P]/(PQ - 1)$ where Q represents E_4 , P represents E_6 , given gradations 4 and 6 modulo 10, respectively. The power series \bar{f} , viewed in this ring has gradation 2 and is equal to $Q^3 - P^2$. [Note that $\Delta = \frac{E_4^3 - E_6^2}{1728}$ and $1728 \equiv 1 \pmod{11}$.]

The power series \bar{f} is also a modular form mod p in the more conceptual sense that it is a section of an appropriate line bundle over a modular curve mod p , e.g., it is a modular form mod p in the sense of Katz.

Such a modular form mod p has a weight that is well-defined only modulo $p - 1$ (i.e., modulo 10 in our particular case). One of the cornerstone results in our theory, that precedes (chronologically) and provides the basis of much of the p -adic interpolation theory that will be one of the principal subjects of this semester, is illustrated by the following theorem. Given the friendly nature of our running numerical example, the theorem has a particularly simple and elegant formulation:

Theorem 2 *For every integer $k \geq 2$ such that $k \equiv 2 \pmod{p - 1} = 10$ there is a (classical) normalized cuspidal modular form f_k on $\Gamma_0(p)$ of weight k ,*

$$f_k = \sum_{n \geq 0} \alpha_n(k) q^n,$$

with Fourier coefficients $\{\alpha_n(k)\}_{n \geq 0}$ generating a ring $\mathcal{O}_k \subset \mathbf{C}$ such that

- *The modular form f_k is an eigenform for all the Hecke operators T_ℓ with $\ell \neq p$ and for the Atkin-Lehner operator U_p ,*

- There is a maximal ideal $m_k \subset \mathcal{O}_k$ with residue field equal to \mathbf{F}_{11} and such that we have the following equation in \mathbf{F}_{11}

$$a_n \equiv \alpha_n(k) \pmod{m_k}$$

for all $n \geq 0$.

- For $k > 2$, the modular form f_k “comes (by the standard mode of augmenting level) from” a classical newform of level 1.
- The completion of \mathcal{O}_k relative to the maximal ideal m_k is \mathbf{Z}_{11} and the modular form f_k is the unique classical eigenform with Fourier coefficients in \mathbf{Z}_{11} on $\Gamma_0(p)$ of weight k whose U_{11} eigenvalue is an 11-adic unit.

For example, f_2 is the classical modular cuspidal (new-) form of weight two on $\Gamma_0(11)$. It has integral Fourier coefficients, and Fourier expansion

$$f_2 = q \prod (1 - q^n)^2 \cdot \prod (1 - q^{11n})^2,$$

while f_{12} is the eigenform on $\Gamma_0(11)$ of slope 0 associated to Δ the (unique) classical newform of weight 12 of level one, and f_{22} is the eigenform on $\Gamma_0(11)$ of slope 0 associated to $E_{10}\Delta = E_4E_6\Delta$ the (unique) classical newform of weight 22 of level one. From now on, let k denote an integer ≥ 2 and congruent to 2 modulo 10.

5 Classical modular forms with coefficients in R

In the discussion of the above theorem, we used the phrase “classical modular form with coefficients in $\mathcal{O} \subset \mathbf{C}$.” It is then time to say what we mean by the adjective “classical” when it comes before “modular form.” So, if M_k is the free \mathbf{Z} -module of finite rank consisting of (truly classical) modular forms of weight $k \geq 2$ on $\Gamma_0(p)$ with Fourier coefficients in \mathbf{Z} , and if R is any commutative ring, by the R -module of “classical modular form with coefficients in R ” (of weight k on $\Gamma_0(p)$) we simply mean:

$$M_k(R) := M_k \otimes R.$$

We keep the adjective “classical” because when $R = \mathbf{Z}_p$ the elements of $M_k(\mathbf{Z}_p)$ are, in general, a much more restricted collection of modular forms than the various brands of p -adic modular forms that will be introduced in due time.

For our series of modular eigenforms

$$f_2, \quad f_{12}, \quad f_{22}, \quad f_{32}, \quad \dots$$

the first three are classical modular forms with coefficients in \mathbf{Z} , but f_{32} (and very likely all the rest) are only “classical modular forms with coefficients in \mathbf{Z}_p ” in the sense that we have just introduced.

6 The Atkin-Lehner operator U_p

[In this section, k can be an arbitrary even positive weight and p doesn't really have to be 11, but no harm if you want to restrict attention to $p = 11$ and $k \equiv 2$ modulo 10.]

Consider this amazingly simple operator on power series:

For R any ring, if $g(q) = \sum_{n \geq 0} c_n q^n \in R[[q]]$ put

$$U_p(g) := \sum_{n \geq 0} c_{pn} q^n \in R[[q]].$$

And so far, there is nothing amazing about this operation. But its beauty is that it preserves classical modular forms of any fixed weight $k \geq 2$ on $\Gamma_0(p)$, in the sense that if $M_k(R)$ is the R -module of classical modular forms of weight $k \geq 2$ on $\Gamma_0(p)$ with Fourier coefficients in R , and $S_k(R) \subset M_k(R)$ is the R -submodule of classical *cuspidal* modular forms of weight $k \geq 2$ on $\Gamma_0(p)$, there is an endomorphism $U_p : M_k(R) \rightarrow M_k(R)$ preserving $S_k(R) \subset M_k(R)$ whose action on Fourier series is given by the formula displayed above. (If R is a field of characteristic 0, then the endomorphism U_p of $M_k(R)$ is a semisimple automorphism of the finite dimension R -vector space $M_k(R)$.) This operator, referred to as the *p*-adic *Atkin-Lehner operator*, plays a dominant role in our theory and is—in some respects—an analogue of the Laplacian operator Δ in the analytic theory. The fact that its effect on Fourier series is given by such simple formula, with no reference to weight in this formula (in contrast to the formulas for all the “other” Hecke operators) is important.

7 U_p where we are back to $p = 11$

Here, let us return to $p = 11$ and $k \equiv 2$ modulo 10. Consider the endomorphism U_p acting on the free, finite rank, \mathbf{Z}_p -module $S_k(\mathbf{Z}_p) = S_k \otimes \mathbf{Z}_p$ (the module of “classical cuspidal modular forms of weight k on $\Gamma_0(p)$ with Fourier coefficients in \mathbf{Z}_p ”). We may decompose $S_k(\mathbf{Z}_p)$ as a (canonical) direct sum of U_p -stable \mathbf{Z}_p -modules,

$$S_k(\mathbf{Z}_p) = S_k(\mathbf{Z}_p)^o \oplus S_k(\mathbf{Z}_p)^{\text{nil}},$$

where U_p acts as an automorphism of $S_k(\mathbf{Z}_p)^o$ and acts topologically nilpotently on $S_k(\mathbf{Z}_p)^{\text{nil}}$. We have, for example, that

$$S_k(\mathbf{Z}_p)^o = \cap_{\nu=1,2,3,\dots} U_p^\nu S_k(\mathbf{Z}_p).$$

In our specific context ($p = 11$ and $k \equiv 2$ modulo 10) $S_k(\mathbf{Z}_p)^o$ is free of rank one and generated by the normalized cuspform f_k .

8 Construction via étale cohomology of Kuga-Sato varieties

All this has some curious bearing on our Galois extension K/\mathbf{Q} . Namely, for each integer $k \geq 2$ we may form the Kuga-Sato variety

$$KS_1^{(k)}(11)$$

(denoted $KS^{(k)}$ for short) which is the $(k-2)$ -fold fiber product of the universal family of elliptic curves (equipped with point of order 11) over the modular curve $X_1(11)$. This $KS^{(k)}$ is an algebraic variety over \mathbf{Q} of dimension $k-1$. We view in its middle dimensional étale cohomology

$$H_{\acute{e}t}^{k-1}(KS_{/\mathbf{Q}}^{(k)}; \mathbf{Z}/11\mathbf{Z})$$

as an \mathbf{F}_{11} -vector space with \mathbf{F}_{11} -linear action of $G_{\mathbf{Q}}$. Within this $G_{\mathbf{Q}}$ -representation space standard operators will allow us to cut out a two-dimensional, irreducible, $G_{\mathbf{Q}}$ -stable subspace,

$$\bar{V}(f_k) \subset H_{\acute{e}t}^{k-1}(KS_{/\mathbf{Q}}^{(k)}; \mathbf{Z}/11\mathbf{Z}),$$

that is (as the literature sometimes says) “attached to” f_k .

The congruence mod 11 displayed in the theorem that is satisfied by the Fourier coefficients of f_k , together with the classical Eichler-Shimura relations guarantees that we have an isomorphism of $G_{\mathbf{Q}}$ -representation spaces

$$\bar{V} \cong \bar{V}(f_k)$$

for all $k \geq 2$. We have therefore a sequence of “realizations” of our extension K/\mathbf{Q} as the splitting field of Galois actions on the irreducible representations $\bar{V}(f_k)$ which occur in the middle dimensional cohomology of a sequence of Kuga-Sato varieties.

9 Liftings to characteristic 0

The $G_{\mathbf{Q}}$ -representation $\bar{V}(f_k)$ is the *residual representation* associated to an irreducible $G_{\mathbf{Q}}$ -representation on a free \mathbf{Z}_{11} -module of rank two

$$V(f_k) \subset H_{\acute{e}t}^{k-1}(KS_{/\mathbf{Q}}^{(k)}; \mathbf{Z}_{11})$$

associated, in a natural manner, to the classical modular eigenform f_k viewed as modular form over the ring \mathbf{Z}_{11} . This representation $V(f_k)$, when restricted to the decomposition group at $p = 11$, has Hodge-Tate weights $(0, k - 1)$.

10 Deformations

From the previous section we see that the single $G_{\mathbf{Q}}$ -representation on the \mathbf{F}_{11} -vector space \bar{V} has an infinity of different liftings to representations on free \mathbf{Z}_{11} -modules of rank two,

$$V(f_2), \quad V(f_{12}), \quad V(f_{22}), \quad \dots$$

each of these liftings being represented in completely different cohomology, with different Hodge numbers. The universal situation of which these liftings are particular instances is the following, but we should duly note that our example residual representation \bar{V} allows for a somewhat more elegant, and simple, answer than does the general case.

Let 11 be denoted p otherwise we'll go dizzy seeing all those 1's, and put $S = \{p, \infty\}$. This will have the consequence that our running letter for prime number is now ℓ .

Let $\Gamma \subset \mathbf{Z}_p^*$ denote the (open) topological group of 1-units, and put $\Lambda := \mathbf{Z}_p[[\Gamma]]$. We have a natural "cyclotomic" character

$$\chi = \chi_p : G_{\mathbf{Q}, S} \rightarrow \mathbf{Z}_p^*$$

which we can project to $\Gamma \subset \Lambda^*$ denoting the ensuing character the *wild cyclotomic character*

$$\chi_{\text{wild}} : G_{\mathbf{Q}, S} \rightarrow \Lambda^*.$$

By

$$\chi_{\text{tame}} : G_{\mathbf{Q}, S} \rightarrow \mathbf{F}_p^* \subset \mathbf{Z}_p^* \subset \Lambda^*$$

we mean the composition of $\bar{\chi}$ with the natural Teichmüller lifting to \mathbf{Z}_p^* and thence to the units in the ring of scalars \mathbf{Z}_p of the completed group ring Λ .

Theorem 3 *There is a continuous representation*

$$\rho : G_{\mathbf{Q}, S} \rightarrow \text{GL}_2(\Lambda)$$

such that

1. *The determinant homomorphism,*

$$\det(\rho) : G_{\mathbf{Q}, S} \rightarrow \text{GL}_2(\Lambda) = \Lambda^*,$$

is the product character

$$\chi_{\text{tame}} \cdot \chi_{\text{wild}} : G_{\mathbf{Q},S} \rightarrow \Lambda^*,$$

2. The restriction of ρ to D_{11} , a decomposition group at 11, has image contained in a subgroup of $\text{GL}_2(\Lambda)$ of the form

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

,

and the restriction of ρ to the corresponding inertia subgroup has image contained in a subgroup of the form

$$\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$$

,

3. Let $m \subset \Lambda$ be the maximal ideal, so that $\Lambda/m = \mathbf{F}_p$. The residual representation obtained by reducing ρ modulo m , i.e., considering the composition

$$\rho : G_{\mathbf{Q}} \rightarrow \text{GL}_2(\Lambda) \rightarrow \text{GL}_2(\Lambda/m) = \text{GL}_2(\mathbf{F}_p),$$

is equivalent to our representation $\bar{\rho}$,

4. The representation ρ is universal with respect to the previous two properties in the sense that if R_0 is a noetherian complete local ring with residue field \mathbf{F}_p and

$$\rho_0 : G_{\mathbf{Q},S} \rightarrow \text{GL}_2(R_0)$$

is a continuous representation such that

- The restriction of ρ_0 to a decomposition group at 11, has image contained in a subgroup of $\text{GL}_2(R_0)$ of the form

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

,

and the restriction of ρ_0 to the corresponding inertia subgroup in D_{11} has image contained in a subgroup of $\text{GL}_2(R_0)$ of the form

$$\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$$

,

- Let $m_0 \subset R_0$ be the maximal ideal of the local ring R_0 , so that $R_0/m_0 = \mathbf{F}_p$. The residual representation obtained by reducing ρ_0 modulo m_0 , i.e., considering the composition,

$$\rho_0 : G_{\mathbf{Q}} \rightarrow \text{GL}_2(R_0) \rightarrow \text{GL}_2(R_0/m_0) = \text{GL}_2(\mathbf{F}_p)$$

is equivalent to our representation $\bar{\rho}$,

then there is a unique local ring homomorphism $\phi_o : \Lambda \rightarrow R_0$ such that the representation ρ_0 is equivalent to the R_0 -representation obtained by composing ρ with the homomorphism

$$\mathrm{GL}_2(\Lambda) \rightarrow \mathrm{GL}_2(R_0)$$

induced from ϕ_o .

This theorem is a difficult theorem to prove, by the way, if you are going to try to do it from scratch. It makes use of Wiles, and Wiles-Taylor, as well as Hida's theory. Note that a specific element of Λ is implicitly defined by the restriction of ρ to the decomposition group at 11: choose a Frobenius element Frob_{11} at 11 (which is only well-defined up to the inertia subgroup of the decomposition group in $G_{\mathbf{Q},S}$) and consider the unramified character of $G_{\mathbf{Q},S}$ implicit in the description of ρ restricted to that decomposition group. Apply this unramified character to Frob_{11} we obtain a well-defined element, $U \in \Lambda$; this element U is congruent to 1 modulo the maximal ideal.

More explicitly, let $D_{11} \subset G_{\mathbf{Q},S}$ be a decomposition group at 11 that is sent to

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

in $\mathrm{GL}_2(\Lambda)$. Choose a Frobenius element $\mathrm{Frob}_{11} \in D_{11}$ and form

$$\rho(\mathrm{Frob}_{11}) = \begin{pmatrix} a & b \\ 0 & u \end{pmatrix}.$$

The “ a ” and the “ b ” may depend upon the choice of Frobenius element at 11 that we have made, but “ u ” does not. The element $u \in \Lambda$ is therefore an *invariant* of the representation ρ that we may immediately see to be the U_p -eigenvalue of the power series F . We have that u is congruent to 1 modulo the maximal ideal of Λ .

11 p -adic modular forms à la Hida

Given our representation ρ we can construct a power series

$$F := \sum_{n \geq 0} T_n q^n \in \Lambda[[q]].$$

Here, for each prime $\ell \neq 11$ we have

$$T_\ell := \mathrm{Trace}_\Lambda(\rho(\mathrm{Frob}_\ell)) \in \Lambda,$$

and $T_{11} := U \in \Lambda$ where U is the element described above. We extend the mapping $\ell \mapsto T_\ell$ for all prime numbers ℓ to a mapping

$$n \mapsto T_n$$

for all $n \geq 0$ using the recursive relations given in section 1.

To make contact with Hida's theory and to preview our discussion of the *eigencurve*, I should remark that the displayed F is an example of a (cuspidal) Λ -adic modular form; we'll see what this means—or at least, we'll see one of the many things that this might mean—momentarily. But first we should talk about the most basic Λ -adic modular form; namely, the Λ -adic *Eisenstein series*.

12 Classical Eisenstein series

Consider the classical family of Eisenstein series E_k for k an even integer (> 2)

$$E_k = 1 - \frac{2k}{b_k} \sum_{n \geq 1} \sigma_{k-1}(n)q^n,$$

where b_k is the k -th Bernoulli number and

$$\sigma_{k-1}(n) := \sum_{0 < d \mid n} d^{k-1},$$

so that

$$E_4 = 1 + 240 \sum_{n \geq 1} \sigma_3(n)q^n,$$

and

$$E_6 = 1 - 504 \sum_{n \geq 1} \sigma_5(n)q^n,$$

and

$$E_{10} = E_4 E_6 = 1 - 11 \cdot 24 \sum_{n \geq 1} \sigma_9(n)q^n \equiv 1 \pmod{11},$$

and more generally

$$E_k \equiv 1 \pmod{11}$$

for k divisible by 10

These power series E_k are the Fourier series of the most basic of Eisenstein series, and are modular eigenforms (for all Hecke operators) of weight k and of level one. This family doesn't yet p -adically interpolate. We need to augment these forms to be eigenforms for $\Gamma_0(p)$.

Put, then,

$$E_k^{(p)}(q) = E_k(q) - p^{k-1} E_k(q^p)$$

giving us eigenforms on $\Gamma_0(p)$ with Fourier expansions

$$E_k^{(p)}(q) = 1 - p^{k-1} - \frac{2k}{b_k} \sum_{n \geq 1} \sigma_{k-1}^{(p)}(n) q^n,$$

where

$$\sigma_{k-1}^{(p)}(n) q^n := \sum_{0 < d \mid n; p \nmid d} d^{k-1},$$

and

$$\mathcal{E}_k^{(p)}(q) = 1 - \frac{2k}{(1 - p^{k-1})b_k} \sum_{n \geq 1} \sigma_{k-1}^{(p)}(n) q^n,$$

which—thanks to the Kummer congruences, and Fermat’s Little theorem— interpolates p -adically, for if $k \equiv k' \pmod{(p-1)p^{\nu-1}}$ we have that

$$\mathcal{E}_k^{(p)}(q) \equiv \mathcal{E}_{k'}^{(p)}(q) \pmod{p^\nu}.$$

Note that

$$U_p(\mathcal{E}^{(p)}) = \mathcal{E}^{(p)}.$$

13 Λ -adic Eisenstein series

Let

$$W := \lim_{\nu} \mathbf{Z}/(p-1)p^{\nu-1}\mathbf{Z}.$$

The even positive numbers, $2\mathbf{N}$, form a dense subset of W . We view W as a *space of p -adic weights*. Later we will think of W as the *classically accessible weights* in the rigid-analytic space \mathcal{W} of \mathbf{C}_p -valued weights (a union of $\frac{p-1}{2}$ disjoint open discs). Any $\kappa \in W$ may also be viewed as a ring homomorphism

$$\text{“}\kappa\text{”} : \Lambda \rightarrow \mathbf{Z}_p$$

characterized by the condition that it sends any group-element $\gamma \in \Gamma \subset \Lambda^*$ to

$$\gamma^\kappa \in \Gamma \subset \mathbf{Z}_p^* \subset \mathbf{Z}_p.$$

Define the **Eisenstein series of weight κ** to be:

$$\mathcal{E}_\kappa^{(p)}(q) = \lim_k \mathcal{E}_k^{(p)}(q) \in \mathbf{Z}_p[[q]]$$

where the limit on the RHS is over positive integers $k \in 2\mathbf{N} \subset W$ that tend to $\kappa \in W$.

Note that since the denominator of b_k is the product of all prime numbers ℓ such that $\ell - 1$ divides $2k$ (*Clausen- Von Staudt*) we get that when $\kappa = 0 \in W$,

$$\mathcal{E}_\kappa^{(p)}(q) = 1.$$

Moreover, our notation in this section and the previous has no inconsistency, for when $\kappa = k \in 2\mathbf{N}$, we have (with the evident interpretation(!))

$$\mathcal{E}_\kappa^{(p)}(q) = \mathcal{E}_k^{(p)}(q).$$

Theorem 4 *There is a power series, $\mathcal{E}^{(p)} \in \Lambda[[q]]$ uniquely characterized by the requirement that “ κ ”(\mathcal{E}) = $\mathcal{E}_\kappa^{(p)}$ for all $\kappa \in W$ such that $\kappa \equiv 0 \pmod{p-1}$.*

[Include a proof of this.]

We can think of $\mathcal{E}^{(p)}$ as a family of p -adic modular forms, parametrized by the weights in W that are congruent to $0 \pmod{p-1}$, and which, for even *integral* weights that are congruent to $0 \pmod{p-1}$, are the classical Eisenstein series of level p . Moreover, $\mathcal{E}^{(p)} \equiv 1 \pmod{p}$.

14 Getting the cuspidal Λ -adic family from the Eisenstein family

We may also view our cuspidal Λ -adic Fourier expansion,

$$F = \sum_{n \geq 0} T_n q^n \in \Lambda[[q]],$$

of the previous section, as a family parametrized by W , setting

$$F_\kappa = \sum_{n \geq 0} \kappa(T_n) q^n \in \mathbf{Z}_p[[q]]$$

for $\kappa \in W$. Here, again, if κ is an even positive integer k congruent to $2 \pmod{p-1}$ then F_κ is a classical modular cuspidal eigenform on $\Gamma_0(p)$ of weight k .

Theorem 5 *(Construction of $F = \sum_{n \geq 0} T_n q^n \in \Lambda[[q]]$)*

$$F = \lim_{\nu \rightarrow \infty} u^{-p^\nu} \cdot U_p^{p^\nu} (f_2 \cdot \mathcal{E}^{(p)}).$$

[Include a proof of this.]

15 Comments on the eigencurve

Our two examples of families

$$\kappa \mapsto \mathcal{E}_\kappa^{(p)}$$

and

$$\kappa \mapsto F_\kappa$$

give us families of power series parametrized, in effect, by an open disc corresponding to weights, that for a dense subset are classical modular eigenforms of a specific level. These then, are our “starting examples” of components of the eigencurve that parametrizes all p -adic modular eigenforms of finite slope for GL_2 over \mathbf{Q} .

Discuss this, its concomitant theories, L -functions (arithmetic and analytic) parametrized by the eigencurve, the skew-Hermitian *organizing modules* and Selmer modules parametrized by the eigencurve [work of Nekovar, Pottharst, Rubin, Skinner-Urban] and Langlands liftings (e.g. symmetric powers) parametrized by the eigencurve.

All this leaves us with the challenge to understand \mathbf{GL}_2 over \mathbf{Q} better and more explicitly [work of Buzzard, Calegari, Emerton, Kisin], and to extend the theory to a more general range of reductive groups, [work of Ash, Buzzard, Chenevier, Coleman, Emerton, Hida, Kisin, Tilouine, Skinner, Urban, . . .] expressing the Langlands program in terms of eigenvarieties. Working with other reductive groups or over fields other than \mathbf{Q} brings up surprising new issues, as in the study of the eigencurve for GL_2 over quadratic imaginary fields [Frank Calegari].

16 A global Langlands correspondence parametrized by the eigencurve? A local Langlands correspondence parametrized by deformations of local Galois representations?

[work of Berger, Breuil, Colmez, Emerton, Kisin, Schneider, Teitelbaum]