

\$(\varphi, \Gamma)\$-MODULES OVER THE ROBBA RING

1. THE ROBBA RING

Let \$L\$ be a finite extension of \$\mathbb{Q}_p\$, and let \$\mathcal{R}_L\$ be the Robba ring with coefficients in \$L\$, *i.e.* the ring of power series

$$f(z) = \sum_{n \in \mathbb{Z}} a_n (z - 1)^n, \quad a_n \in L$$

converging on some annulus of \$\mathbb{C}_p\$ of the form \$r(f) \leq |z - 1| < 1\$, equipped with its natural \$L\$-algebra topology. It is a domain, but it is not noetherian.

Theorem 1 (Lazard, see e.g. [Be1] prop. 4.12).

- (i) *Any finitely generated ideal of \$\mathcal{R}_L\$ is principal.*
- (ii) *Any finite type submodule of \$\mathcal{R}_L^n\$ admits elementary divisors.*

Remark 1. Part (i) implies that finite type, torsion free, \$\mathcal{R}_L\$-modules are free. Part (ii) implies e.g. that if \$M \subset N := \mathcal{R}_L^n\$ is of finite type over \$\mathcal{R}_L\$, then the saturation of \$M\$ in \$N\$ (that is \$M^{\text{sat}} = \{x \in N, \exists f \neq 0 \in \mathcal{R}_L, fx \in M\}\$) is finite type over \$\mathcal{R}_L\$ with the same rank as \$M\$.

The ring \$\mathcal{R}_L\$ is equipped with commuting, \$L\$-linear, continuous actions of \$\varphi\$ and \$\Gamma := \mathbb{Z}_p^*\$ defined by

$$\varphi(f)(z) = f(z^p), \quad \gamma(f)(z) = f(z^\gamma).$$

(note that here \$z \in \mathbb{C}_p\$ satisfies \$|z - 1| < 1\$). Set

$$t := \log(z) := \sum_{n \in \mathbb{Z}_{\geq 1}} (-1)^{n+1} \frac{(z - 1)^n}{n} \in \mathcal{R}_L.$$

Then \$\varphi(t) = pt\$ and \$\gamma(t) = \gamma t\$.

Lemma 1 ([Co2] rem. 4.4). *The finitely generated ideals of \$\mathcal{R}_L\$ stable by \$\varphi\$ and \$\Gamma\$ are the \$t^i \mathcal{R}_L\$, \$i \ge 0\$ an integer.*

2. \$(\varphi, \Gamma)\$-MODULES OVER \$\mathcal{R}_L\$

Definition 1. A \$(\varphi, \Gamma)\$-module over \$\mathcal{R}_L\$ is a finite free \$\mathcal{R}_L\$-module \$D\$ equipped with commuting, \$\mathcal{R}_L\$-semilinear, continuous¹ actions of \$\varphi\$ and \$\Gamma\$, and such that \$\mathcal{R}\varphi(D) = D\$.

Works of Fontaine, Cherbonnier-Colmez, and Kedlaya, allow to define a \$\otimes\$-equivalence \$D_{\text{rig}}\$ between the category of \$L\$-representations of \$\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)\$ and \$\textit{étale}\$ \$(\varphi, \Gamma)\$-modules over \$\mathcal{R}_L\$. By [Be1, §3.4], \$D_{\text{rig}}(V)\$ can be defined in Fontaine's style:

¹It means that for any choice of a free basis \$e = (e_i)_{i=1 \dots d}\$ of \$D\$ as \$\mathcal{R}_L\$-module, the matrix map \$\gamma \mapsto M_e(\gamma) \in \text{GL}_d(\mathcal{R}_L)\$, defined by \$\gamma(e_i) = M_e(\gamma)(e_i)\$, is a continuous function on \$\Gamma\$. If \$P \in \text{GL}_d(\mathcal{R}_L)\$, then \$M_{P(e)}(\gamma) = \gamma(P)M_e(\gamma)P^{-1}\$, hence it suffices to check it for a single basis.

there exists a topological ring B (denoted $B^{\dagger, \text{rig}}$ there) equipped with actions of $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ and φ and such that $B^{H_p} = \mathcal{R}$, and

$$D_{\text{rig}}(V) := (V \otimes_{\mathbb{Q}_p} B)^{H_p}.$$

Here, H_p is the kernel of the cyclotomic character $\chi : \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \rightarrow \mathbb{Z}_p^*$, inducing an isomorphism $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)/H_p \xrightarrow{\sim} \Gamma$.

Proposition 1. ([Co2, prop. 2.7]) *The functor D_{rig} induces an \otimes -equivalence of category between finite dimensional, continuous, L -representations of $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ and étale (φ, Γ) -modules over \mathcal{R}_L . We have $\text{rk}_L(V) = \text{rk}_{\mathcal{R}_L}(D_{\text{rig}}(V))$.*

Remark 2. A (φ, Γ) -module is étale if its underlying φ -module has slope 0 in the sense of Kedlaya (see [Ke1, Theorem 6.10] or [Co2, §2.1]). Kedlaya defines some notion of slopes for φ -modules over \mathcal{R}_L (such that $\varphi(M)\mathcal{R}_L = M$) and proves that any such module has a canonical filtration by isoclinic φ -submodules whose slopes are strictly increasing ([Ke1, Theorem 6.10]). In the (φ, Γ) -module situation, this φ filtration turns out to be stable by Γ (see [Be2] part IV).

3. (φ, Γ) -MODULES OF RANK 1 AND THEIR EXTENSIONS, FOLLOWING COLMEZ.

Let $\delta : \mathbb{Q}_p^* \rightarrow L^*$ be a continuous character. Colmez defines in [Co2, §0.1], the (φ, Γ) -module $\mathcal{R}_L(\delta)$ which is \mathcal{R}_L as \mathcal{R}_L -module but equipped with the \mathcal{R}_L -semilinear actions of φ and Γ defined by

$$\varphi(1) := \delta(p), \quad \gamma(1) := \delta(\gamma), \quad \forall \gamma \in \Gamma,$$

Recall that by class field theory the cyclotomic character χ extends uniquely to an isomorphism $\theta : W_{\mathbb{Q}_p}^{\text{ab}} \xrightarrow{\sim} \mathbb{Q}_p^*$ sending the geometric Frobenius to p , where $W_{\mathbb{Q}_p} \subset \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ is the Weil group of \mathbb{Q}_p . We may then view any δ as above as a continuous homomorphism $W_{\mathbb{Q}_p} \rightarrow L^*$. Such a homomorphism extends continuously to $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ iff $v(\delta(p))$ is zero, and in this case we see that

$$\mathcal{R}_L(\delta) = D_{\text{rig}}(\delta \circ \theta).$$

Proposition 2. [Co2, Thm 0.2]

- (i) *Any (φ, Γ) -module free of rank 1 over \mathcal{R}_L is isomorphic to $\mathcal{R}_L(\delta)$ for a unique δ . Such a module is isocline of slope $v(\delta(p))$.*
- (ii) *$\text{Ext}_{(\varphi, \Gamma)}(\mathcal{R}_L(\delta_2), \mathcal{R}_L(\delta_1))$ has L -dimension 1 except when $\delta_1\delta_2^{-1} = x^{-i}$ or $\chi \cdot x^i$ for $i \geq 0$ an integer, in which case it has dimension 2.*

Here, $x : \mathbb{Q}_p^* \rightarrow L^*$ is the inclusion, and $\chi = x|x|$ is the character such that $\chi(p) = 1$ and $\chi|_{\Gamma} = x|_{\Gamma}$ is the natural inclusion. Colmez computes also Kedlaya's slopes of such extensions (see Rem. 0.3 of [Co2]). An important fact is that the extension can be étale (hence coming from a p -adic representation) even if the $\mathcal{R}_L(\delta_i)$'s are not. Some necessary conditions of etaleness are that $v(\delta_1(p)) \geq 0$ and $v(\delta_1(p)\delta_2(p)) = 0$ (étalness of the determinant), these conditions are also sufficient in most cases (but see *loc. cit.*).

Definition 2. Let D be a (φ, Γ) -module of rank d over \mathcal{R}_L and equipped with a strictly increasing filtration $(\text{Fil}_i(D))_{i=0\dots d}$:

$$\text{Fil}_0(D) := \{0\} \subsetneq \text{Fil}_1(D) \subsetneq \dots \subsetneq \text{Fil}_i(D) \subsetneq \dots \subsetneq \text{Fil}_{d-1}(D) \subsetneq \text{Fil}_d(D) := D,$$

of (φ, Γ) -submodules which are free and direct summand as \mathcal{R}_L -modules. We call such a D a *triangular* (φ, Γ) -module over \mathcal{R}_L , and the filtration $\mathcal{T} := (\text{Fil}_i(D))$ a *triangulation of D over \mathcal{R}_L* .

Following Colmez, we shall say that a (φ, Γ) -module which is free of rank d over \mathcal{R}_L is *triangulable* if it can be equipped with a triangulation \mathcal{T} ; we shall say that an L -representation V of $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ is *trianguline* if $D_{\text{rig}}(V)$ is triangulable.

Let D be a triangular (φ, Γ) -module. By prop. 2, each

$$\text{gr}_i(D) := \text{Fil}_i(D)/\text{Fil}_{i-1}(D), \quad 1 \leq i \leq d,$$

is isomorphic to $\mathcal{R}_L(\delta_i)$ for some unique $\delta_i : W_{\mathbb{Q}_p} \rightarrow L^*$. It makes then sense to define the *parameter of the triangulation* to be the continuous homomorphism

$$\delta := (\delta_i)_{i=1, \dots, d} : \mathbb{Q}_p^* \rightarrow (L^*)^d.$$

4. p -ADIC HODGE THEORY OF (φ, Γ) -MODULES, FOLLOWING BERGER.

Let D be a fixed (φ, Γ) -module. When $D = D_{\text{rig}}(V)$ for some p -adic representation V , D uniquely determines V hence it makes sense to ask whether we can directly recover from D the usual Fontaine's functors of V . The answer is yes and achieved by Berger's work ([Be1], [Be2]). It turns out that it makes sense to define these Fontaine functors for any (φ, Γ) -module over \mathcal{R}_L (i.e. not necessarily étale). In what follows, we may and do assume that $L = \mathbb{Q}_p$, $\mathcal{R} := \mathcal{R}_{\mathbb{Q}_p}$.

Let us introduce, for $r > 0 \in \mathbb{Q}$, the \mathbb{Q}_p -subalgebra

$$\mathcal{R}_r = \{f(z) \in \mathcal{R}, f \text{ converges on the annulus } p^{-\frac{1}{r}} \leq |z-1| < 1\}.$$

Note that \mathcal{R}_r is stable by Γ , and that φ induces a map $\mathcal{R}_r \rightarrow \mathcal{R}_{pr}$ when $r > \frac{p-1}{p}$ which is étale of degree p . The following lemma is [Be2, thm 1.3.3]:

Lemma 2. *Let D be a (φ, Γ) -module over \mathcal{R} . There exists a $r(D) > \frac{p-1}{p}$ such that for each $r > r(D)$, there exists a unique finite free, Γ -stable, \mathcal{R}_r -submodule D_r of D such that $\mathcal{R} \otimes_{\mathcal{R}_r} D_r \xrightarrow{\sim} D$ and that $\mathcal{R}_{pr} D_r$ has a \mathcal{R}_{pr} -basis in $\varphi(D_r)$. In particular, for $r > r(D)$,*

- (i) for $s \geq r$, $D_s = \mathcal{R}_s D_r \xrightarrow{\sim} \mathcal{R}_s \otimes_{\mathcal{R}_r} D_r$,
- (ii) φ induces an isomorphism $\mathcal{R}_{pr} \otimes_{\mathcal{R}_r, \varphi} D_r \xrightarrow{\sim} D_{pr} \xrightarrow{\sim} \mathcal{R}_{pr} \otimes_{\mathcal{R}_r} D_r$.

If $n(r)$ is the smallest integer n such that $p^{n-1}(p-1) \geq r$, then for $n \geq n(r)$ the primitive p^n -th roots of unity lie in the annuli $p^{-\frac{1}{r}} < |z-1| < 1$ and t is a uniformizer at each of them so that we get by localization and completion at their underlying closed point a natural map

$$\mathcal{R}_r \rightarrow K_n[[t]], \quad n \geq n(r), \quad r > r(D),$$

which is injective with t -adically dense image, where $K_n := \mathbb{Q}_p(\sqrt[p^n]{1})$. For any (φ, Γ) -module over \mathcal{R} , we can then form for $r > r(D)$ and $n \geq n(r)$ the space

$$D_r \otimes_{\mathcal{R}_r} K_n[[t]],$$

which is a $K_n[[t]]$ -module free of rank $\text{rk}_{\mathcal{R}}(D)$ equipped with a semi-linear continuous action of Γ . By Lemma 2 (i), this space does not depend on the choice of r

such that $n \geq n(r)$. Moreover, for a fixed r , φ induces by the same lemma part (ii) – $\otimes_{\mathcal{R}_{pr}} K_{n+1}[[t]]$ a Γ -equivariant, $K_{n+1}[[t]]$ -linear, isomorphism

$$(D_r \otimes_{\mathcal{R}_r} K_n[[t]]) \otimes_{t \mapsto pt} K_{n+1}[[t]] \longrightarrow D_r \otimes_{\mathcal{R}_r} K_{n+1}[[t]].$$

(Note that the map $\varphi : \mathcal{R}_r \longrightarrow \mathcal{R}_{pr}$ induces the inclusion $K_n[[t]] \longrightarrow K_{n+1}[[t]]$ such that $t \mapsto pt$.)

We use this to define functors $\mathcal{D}_{\text{Sen}}(D)$ and $\mathcal{D}_{\text{dR}}(D)$, as follows. Let $K_\infty = \bigcup_{n \geq 0} K_n$. For $n \geq n(r)$ and $r > r(D)$, we define a K_∞ -vector space with a semi-linear action of Γ by setting

$$\mathcal{D}_{\text{Sen}}(D) := (D_r \otimes_{\mathcal{R}_r} K_n) \otimes_{K_n} K_\infty.$$

By the discussion above, this space does not depend of the choice of n, r . In the same way, the \mathbb{Q}_p -vector spaces

$$\mathcal{D}_{\text{dR}}(D) := (K_\infty \otimes_{K_n} K_n((t)) \otimes_{\mathcal{R}_r} D_r)^\Gamma,$$

$$\text{Fil}^i(\mathcal{D}_{\text{dR}}(D)) := (K_\infty \otimes_{K_n} t^i K_n[[t]] \otimes_{\mathcal{R}_r} D_r)^\Gamma \subset \mathcal{D}_{\text{dR}}(D), \quad \forall i \in \mathbb{Z},$$

are independent of $n \geq n(r)$ and $r > r(D)$. As $K_\infty((t))^\Gamma = \mathbb{Q}_p$, $\mathcal{D}_{\text{dR}}(D)$ so defined is a finite dimensional \mathbb{Q}_p -vector-space whose dimension is less than $\text{rk}_{\mathcal{R}}(D)$, and $(\text{Fil}^i(\mathcal{D}_{\text{dR}}(D)))_{i \in \mathbb{Z}}$ is a decreasing, exhausting, and saturated, filtration on $\mathcal{D}_{\text{dR}}(D)$.

We end by the definition of $\mathcal{D}_{\text{crys}}(D)$. Let

$$\mathcal{D}_{\text{crys}}(D) := D[1/t]^\Gamma.$$

It has an action of $\mathbb{Q}_p[\varphi]$ induced by the one on $D[1/t]$. It has also a natural filtration defined as follows. Choose $r > r(D)$ and $n \geq n(r)$, there is a natural inclusion

$$\mathcal{D}_{\text{crys}}(D) \longrightarrow \mathcal{D}_{\text{dR}}(D)$$

and we denote by $(\varphi^n(\text{Fil}^i(\mathcal{D}_{\text{crys}}(D))))_{i \in \mathbb{Z}}$ the filtration induced from the one on $\mathcal{D}_{\text{dR}}(D)$. By the analysis above, this defines a unique filtration $(\text{Fil}^i(\mathcal{D}_{\text{crys}}(V)))_{i \in \mathbb{Z}}$, independent of the above choices of n and r . We summarize some of Berger's results ([Be1, thm. 0.2 and §5.3], [Be2], [Co1, prop. 5.6]) in the following proposition.

Proposition 3. *Let V be a \mathbb{Q}_p -representation of $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$, and*

$$* \in \{\text{crys}, \text{dR}, \text{Sen}\}.$$

Then $\mathcal{D}_(D_{\text{rig}}(V))$ is canonically isomorphic to $\mathcal{D}_*(V)$.*

Definition 3. We will say that a (not necessarily étale) (φ, Γ) -module D over \mathcal{R} is crystalline (resp. de Rham) if $\mathcal{D}_{\text{crys}}(D)$ (resp. $\mathcal{D}_{\text{dR}}(D)$) has rank $\text{rk}_{\mathcal{R}}(D)$ over \mathbb{Q}_p . The Sen polynomial of D is the one of the semi-linear Γ -module $\mathcal{D}_{\text{Sen}}(D)$.

Here is an example of application to triangular (φ, Γ) -modules. Let D be a triangular (φ, Γ) -module of rank d over \mathcal{R}_L , whose parameter is $(\delta_i)_{i=1, \dots, d}$. Define the *weight* $\omega(\delta) \in L$ of any continuous character $\delta : \mathbb{Q}_p^* \longrightarrow L^*$ by the formula

$$\omega(\delta) := - \left(\frac{\partial \delta|_\Gamma}{\partial \gamma} \right)_{\gamma=1} = - \frac{\log(\delta(1+p^2))}{\log(1+p^2)} \in L.$$

Proposition 4. [BCh, prop. 2.3.3, 2.3.4] *Let D be a triangular (φ, Γ) -module over \mathcal{R}_L with parameter $(\delta_i)_{i=1, \dots, d}$.*

- (i) *The Sen polynomial of $D_{\text{Sen}}(D)$ is $\prod_{i=1}^d (T - \omega(\delta_i))$.*
- (ii) *Assume that each $\omega(\delta_i) \in \mathbb{Z}$ and that the sequence $\omega(\delta_1), \omega(\delta_2), \dots, \omega(\delta_d)$ is strictly increasing. Then D is de Rham.*

REFERENCES

- [BCh] J. Bellaïche, G. Chenevier, *p-adic families of Galois representations*, chapter 2, preprint (see my home page).
- [Be1] L. Berger, *Représentations p-adiques et équations différentielles*, Inv. Math. 148 (2002), 219–284.
- [Be2] L. Berger, *Équations différentielles p-adiques et (φ, N) -modules filtrés*, preprint (see his homepage).
- [Co1] P. Colmez, *Les conjectures de monodromie p-adiques*, in *Sém. Bourbaki 2001-02*, exp. 897, Astérisque 290 (2003), 53–101.
- [Co2] P. Colmez, *Série principale unitaire pour $GL_2(\mathbb{Q}_p)$ et représentations triangulines de dimension 2*, preprint.
- [Ke1] K. Kedlaya, *A p-adic local monodromy theorem*, Annals of Mathematics 160 (2004), 93–184.