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QR 26: Choice and Chance
The Mathematics of Decision Making

Unit I Exercises

I.A Problem #2 from the handout "Measurement Scales" by Jo Anne Growney

I.B Problem #8 from the hand out "Measurement Scales" by Jo Anne Growney

I.C Problem #3 from the handout "Measurement Scales" by Jo Anne Growney

I.D Serving on national fellowship selection committee, you notice that the system for scaling letter grades to compute class averages is different at Harvard from other colleges. If student record *x* ranks higher than *y* at Harvard, can you conclude that the same grade record *x* would also rank higher than *y* at other colleges? Do it make sense to say Record *x* is 10% better than record *y*?   Record *x* is twice as strong as Record *y*?   Does it make a difference if the two averages are computed on the same or different scales?   By what system would you rank students based on their grades and with what reservations? (cf. Activity I)

I.E Starting from the assumption that both are interval scales, derive formulae for switching between Fahrenheit and Centigrade by imposing what you know about the freezing and boiling points of water in each system. Show that, in contrast to the previous example, the question of whether the average temperature (as measured each day at noon, say) in City X is bigger than in City Yover a given period does not depend on which scale you use in both.

I.F Do at least one problem from the Kreps Chapters in the Sourcebook. This material is pretty technical. Just make sure you understand what a binary relation is, what properties it can have, what properties it must have to be called a preference relation, and when such a relation determines a choice function.

I.G You are decorating a room for an indecisive grandparent, who will not tell you what color paint to use, but will give you a strict and consistent preference between any pair of colors. The only choices are Lilac, Aqua, Brown, Mushroom, White, Green, and Pink. Using first letters as abbreviations, you have asked seven questions and learned that your grandparent holds:

$M>P$, $M>G$, $P>W$, $A>M$, $G>W$, $B>L$,   and $L>M$.

What is the next and last question you ask in order to determine what color to paint the room? In all, you have asked eight questions. What is the least number you could have asked, and is there a strategy for questioning that ensures you only ask the least number possible? Give reasons for your answer.   [F]

I.H You are a judge in an amateur pie baking contest and must pick a winner from among four entries labeled *x, y, z,* and *w*. You rank each according to seven attributes, say Appearance, Calories, Flavor, Novelty, Smell, Texture, and Vitamins. The highest ranked pie in each category gets 4 points, the next highest 3, etc. The one with the highest number of points when added all together wins. This method is sometimes called a Borda count. Here is the data:

	A	C	F	N	S	T	V	Total
X	4	1	2	4	2	2	4	18
Y	3	4	1	3	4	1	3	19
Z	2	3	4	2	3	4	2	20
W	1	2	3	1	2	3	1	13

i. After *z* is declared the winner, it is discovered that *w* should have been disqualified because it was baked by a professional. This should not matter, it seems, since *z* was found better than *w* in every category. But would *z* have won if there were only three choices to begin with so that the top ranked got 3 points and so on?

ii. Referring to Chapter 3 of Kreps, what would Sen say this means about the choice function *c(.)* that selects the Borda winner out of each subset of {*x, y, z, w*} set? Are there preferences < on that set such that the Borda choice function is of the form c(., <) in the notation of Kreps, Chapter 3, meaning that the choice function chooses the best alternatives according to these preferences?

iii. What do you think is fair to do about awarding the pie baking prize? Do you think it becomes easier or harder to pick a winner fairly if the judge is allowed to rate rather than just rank the entries according to each attribute?

Mathematical challenge: When doing a Borda count like this with *n* alternatives, each column must add up to 1+2+3+…+*n*. Find and justify a formula for this sum. Look up the legend about how the mathematician Gauss figured out the answer as a schoolboy.

Research challenge: Find out to what extent the Borda count describes how winners are picked in Consumer Reports, in Cambridge elections, in the Heisman Trophy competition, etc.

I.I Optional Mathematical Exercise

Notation

Elements will be designated by small letters: w,x,y,z
We assume there are finitely many elements (interpreted as action alternatives).
Sets of elements will be designated by capital letters: A, B, C,
The symbol {x,y,z} will designate the set consisting of elements x, y, z.

To talk about the best alternatives in a set, we introduce the function **B** that operates on sets to produce the best of that set. We allow for ties so for a given set V, the set **B**(V) may consist of more than one element**.**

Assumed Properties of the B function.

1. For any set V, **B**(V) is not empty. This implies all elements are comparable.

2. **B**(V)   V, for any V .

Conditions 1 and 2 imply that for any element x ,

B({x}) = {x}.   (Why?)   (1)

3. **Independence of Irrelevant Alternative (IIA) Condition**

For any sets Vand W,

B (V   W) = **B** [**B**(V)   B(W)].   (2)

Questions:

a. Interpret the meaning of Condition 3.

b. Show that if element x belongs to V but not to **B**(V), then x does not belong to **B** (V   {y}); i.e., you should not be able to turn a non-best element in a set into a best element by enlarging the set.

c. Show that if x belongs to V but not to **B(**V), then **B**(V - {x}) = **B**(V), i.e., deleting a non-best element from V does not change what's best in the depleted set.

The Preference Relations P and I associated with B.

Definition : For any two-element set {x, y}, let

xPy   if and only if   **B**{x, y} = {x} ;   (3a)

and let

xIy   if and only if   **B**{x, y} = {x, y} .   (3b)

We say P is the preference binary relation associated with **B.** Read xPy as "x is preferred to y." Notice that **B**({x, y}) = { x }, states a preference for x over y.

d. Prove the following theorem.

Theorem: If **B** satisfies conditions 1, 2, and 3, then its associated preference relation P is transitive. Hint: Suppose not, etcetera.