QR 26: Choice and Chance
The Mathematics of Decision Making

Solutions to Unit III Exercises

References:

(a) [HH] Harnett and Horrell, *Data, Statistics, and Decision Models with Excel*.
(b) [DSS] Dorfman, Samuelson, and Solow, *Linear Programming and Economic Analysis*.
(c) [BHM] Bradley, Hax, and Magnanti, *Applied Mathematical Programming*.
(d) [R1] Rosenberg, N. UMAP Module 453: "Linear Programming in Two Dimensions: I"
(e) [R2] Rosenberg, N. UMAP Module 454: "Linear Programming in Two Dimensions: II"

III.A. *Set up problems 1–2 of the handout, UMAP Module 453 [R1]. You need not carry out the optimization procedure, but do identify the choice variables, the constraints, and the objective function.*

Exercise 1. *A dry cleaning company is buying up to 30 new pressing machines and is considering both a deluxe and a standard model. The deluxe model occupies 2 square yards of floor space and presses 3 pieces per minute. The standard model occupies 1 square yard of floor space but presses only 2 pieces per minute. If 44 square yards of floor space are available, how many machines of each type should the company buy to maximize its output?*

Let x be the number of deluxe models purchased and y the number of standard models purchased. Then the objective function is

$$\text{speed} = 3x + 2y \text{ pieces pressed/minute}$$

which we want to maximize. The constraints are:

$$2x + y \le 44 \text{ sq. yds. (floor space)},$$
$$x + y \le 30 \text{ (\# machines purchased)},$$

and, of course, $x \ge 0$ and $y \ge 0$. These constraints shall be implicit throughout for any linear programming problem.

Exercise 2. *The producers of a Broadway musical plan to advertise on New York City buses and on a local radio station. Each bus advertisement costs \$1000; each radio commercial costs \$3000. The producers want to have at least one-third as many bus advertisements as radio commercials. Bus advertisements are known to reach 400 upper income families, 400 middle income families, and 500 lower income families each week. The radio commercials reach 100 upper income families, 1100 middle income families, and 100 lower income families each week. If the producers want to reach at least 2100 upper income families and 9100 middle income families and no more than 5000 lower*

income families every week, how should they distribute their advertising between the two media in order to minimize the cost of their campaign?

Let x be the number of bus advertisements and y the number of radio commercials purchased per week. Then the objective is to minimize

$$\text{cost} = 1000x + 3000y \text{ dollars}$$

subject to the constraints:

$$x \quad 1/3y \text{ (proportionality constraint)}$$
$$400x + 100y \quad 2100 \text{ (upper income families reached per week)}$$
$$400x + 1100y \quad 9100 \text{ (middle income families reached per week)}$$
$$500x + 100y \quad 5000 \text{ (lower income families reached per week)}$$

III.B. *Exercise 3 of the hand-out, UMAP module 454 [R2], "Linear Programming in Two Dimensions, II." The farmer (Exercises 3 and 7, Linear Programming I and pages 1, 2, and 3 of this module) finds that he can get a short-term loan at 15% interest. What is the shadow price of capital? Will it pay the farmer to borrow?*

We set up the problem on Microsoft Excel like this:

	A	B	C	D	E	F
1		**Old Macdonald Produce Co.**				
2		We've got more than just Oink-Oink!				
3		Vegetable Mix Analysis				
4						
5	**Decision Variables**	**Tomatoes**	**Corn**			
6	Values (bushels)	13	12			
7	Profit	$ 2.00	$ 3.00	$ 62.00		
8						
9	**Constraints**					**Bound**
10	Water (gallons)	10	60	850	<=	30,000
11	Land (acres)	0.05	0.025	0.95	<=	30
12	Capital	1	1	$ 25.00	<=	750
13	Demand on Tomatoes	1		13	<=	500
14	Demand on Corn		1	12	<=	475

Of course, the cell D7 has the formula =SUMPRODUCT($B$6:$C$6,B7:C7) in it. This is then copied all the way down into cells D10:D14.

According the to Solver, the optimal choice is to get 300 bushels of tomatoes and 450 bushels of corn. Reading the Answer and Sensitivity reports, we find that the capital constraint is binding with a shadow price of $1.80. Since at 15% interest it costs $1.15 later to get an additional $1 in capital, the farmer should take the loan. But the allowable

increase for this constraint is only $68.18, so borrowing any more than that would move the optimal solution.

III.C. [S] *Your boat company makes four different kinds of boats: Large sailboats (at $1200 profit per boat), small sailboats (@ $930), motorboats (@ $1050) and sailboards (@ $750). Each boat requires some of your raw materials on hand, according to the table that follows.*

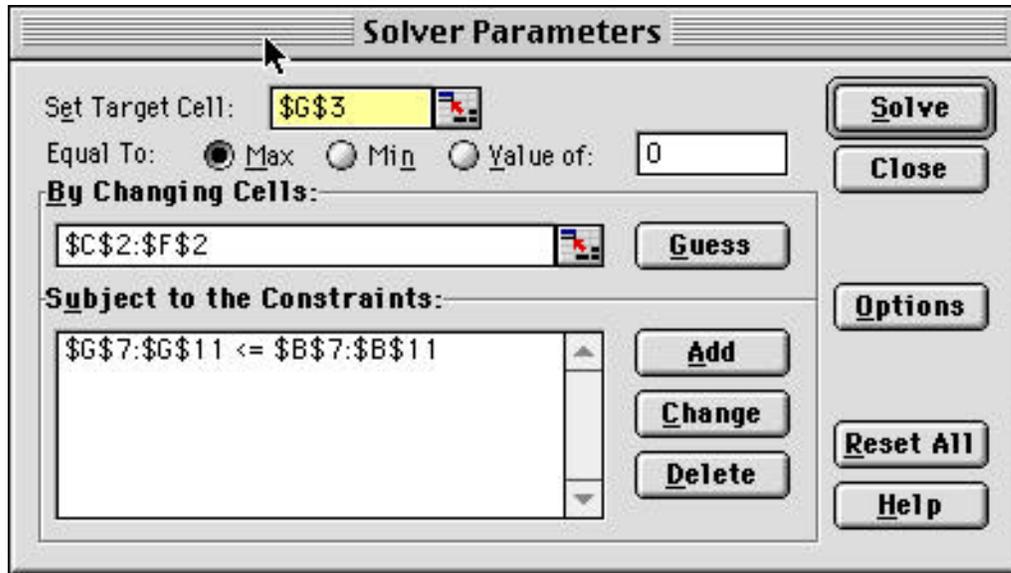| Raw Materials	On Hand	Requirements by Product			
		Large Sailboat	Small Sailboat	Motorboat	Sailboard
Sailcloth	700	4	3	0	1
Glass Fiber	1,380	8	3	4	2
Epoxy Resin	1,280	3	3	3	2
Aluminum	1,100	4	2	2	2
Engines	120	0	0	1	0

Manufacturing what mix of products using the resources on hand will generate the highest profit? Which raw material would you most like more of and why?

Here is a Microsoft Excel spreadsheet describing the problem. You ought to be able to guess what's in the cells G3 and G7:G11.

	A	B	C	D	E	F	G
1	Boat Types		Large Sailboat	Small Sailboat	Motorboat	Sailboard	
2	Number Made		0	0	0	0	Total Profit
3	Profit by Product		$ 1,200.00	$ 930.00	$ 1,050.00	$ 750.00	$ -
4							
5			Requirements by Product				
6	Raw Materials	On Hand	Large Sailboat	Small Sailboat	Motorboat	Sailboard	Total Materials Used
7	Sailcloth	700	4	3	0	1	0
8	Glass Fiber	1,380	8	3	4	2	0
9	Epoxy Resin	1,280	3	3	3	2	0
10	Aluminum	1,100	4	2	2	2	0
11	Engines	120	0	0	1	0	0

We've also shown a picture of the Solver dialog box. Solver instructs us to build no large sailboats, 80 small sailboats, 100 motorboats, and 370 sailboards.

Excerpting from the Answer and Sensitivity Reports, we see that three of the constraints are binding: the glass fiber, the epoxy resin, and the aluminum. Of these, the aluminum has the highest shadow price of $195, meaning that each additional unit of aluminum will get us this much more profit.

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Solver Parameters

Set Target Cell: $G$3
Equal To: ● Max ○ Min ○ Value of: 0
By Changing Cells:
$C$2:$F$2   Guess
Subject to the Constraints:
$G$7:$G$11 <= $B$7:$B$11   Add   Change   Delete

Solve   Close   Options   Reset All   Help

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$G$7	Sailcloth Total Materials Used	610	$G$7<=$B$7	Not Binding	89.99999995
$G$8	Glass Fiber Total Materials Used	1380	$G$8<=$B$8	Binding	0
$G$9	Epoxy Resin Total Materials Used	1280	$G$9<=$B$9	Binding	0
$G$10	Aluminum Total Materials Used	1100	$G$10<=$B$10	Binding	0
$G$11	Engines Total Materials Used	99.99999999	$G$11<=$B$11	Not Binding	20.00000001

Constraints

Cell	Name	Final Value	Shadow Price
$G$7	Sailcloth Total Materials Used	610	0
$G$8	Glass Fiber Total Materials Used	1380	120
$G$9	Epoxy Resin Total Materials Used	1280	60.00000002
$G$10	Aluminum Total Materials Used	1100	195
$G$11	Engines Total Materials Used	99.99999999	0

III.D. *Exercises 12.5–12.6 from [HH]. Bank Cashy has only two types of assets, loans (x) and investments (y). A total of $700 million is to be allocated between x and y. Bank Cashy wishes its loans to equal at least $300 million and its investments to be at least 30% of the total of x + y. The bank earns 12% on its loans and 14% on its investments*
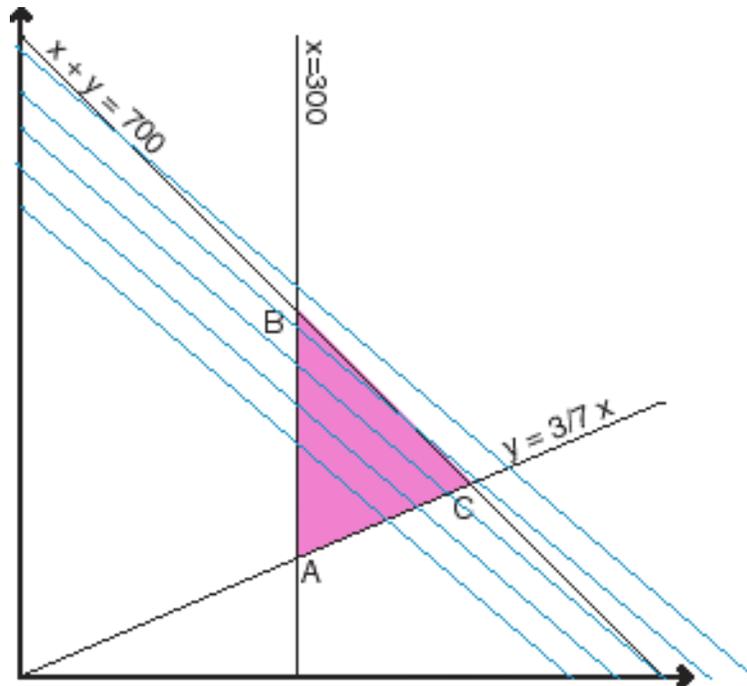> (a) Formulate the LP problem to maximize profit.
> (b) Find the optimal solution graphically.
> (c) Use Solver to find the optimal solution.

(a) The function we are trying to maximize is

$$\text{profit} = 0.12x + 0.14y.$$

One constraint is easy: $x \geq 300$ (we'll work in units of millions of dollars). For the second, we need $x \geq 3/10(x + y)$. Manipulating this inequality gives $y \leq (3/7)x$.

(b)

Here the pink region is the feasible set, and the blue lines are level curves of the objective function. We test the vertices A, B, C.

Vertex	Coordinates	Profit
A	(300, 900/7)	$(300)(12/100)+ (900/7)(14/100) = 36 + 18 = 54$
B	(300, 400)	$(300)(12/100) + (400)(14/100) = 36 + 56 = 92$
C	(490, 210)	$(490)(12/100) + (210)(14/100) = 88.2$

Apparently, B is the point of maximal profit.

(c) Below is the Excel Spreadsheet. By now you should know what to do. Notice that we rewrote the constraint $y \leq 1/3$ x as $3x - y \geq 0$.

Use a graph and a sensitivity report to answer the following questions:
  (a) *What is the Shadow Price for the constraint that loans are $\geq$ $300million? By how much can this right-hand side change before the corner point increases? What is the Shadow Price for the constraints that investments are $\leq$ 30%?*
 (b) *Show graphically the effect of a change in the return on investments from 14% to 10%.*

(a) We refer to the Solver report. Apparently, the Shadow Price for the minimum loans constraint is -$.02! What does that mean? Increasing the right-hand side of this

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constraint means that we must keep more of our money in the lower-interest loans than in investments. So if we to reduce this constraint by $1 million, we would end up with $20,000 more profit.

	A	B	C	D	E	F
1			Bank Cashy			
2			Allocation Plan			
3		Loans	Investments			
4	Amounts	$ 100.00	$ 400.00			
5	Return	12%	14%	$ 68.00		
6						
7	Constraints					
8	Min. Loans	1		$ 100.00	>=	300
9	Proportionality	3	-1	$ (100.00)	>=	0
10	Capital	1	1	$ 500.00	<=	750

The Shadow Price for the proportion of loans is zero, because it's not a binding constraint.

(b) Graphically, the isoprofit line through vertex B changes slope when the rate of return on investments decreases. Now, instead of keeping the feasibility set all on one side, it "pierces" the set at B. This means the optimal corner has pivoted to another vertex, in this case, C.
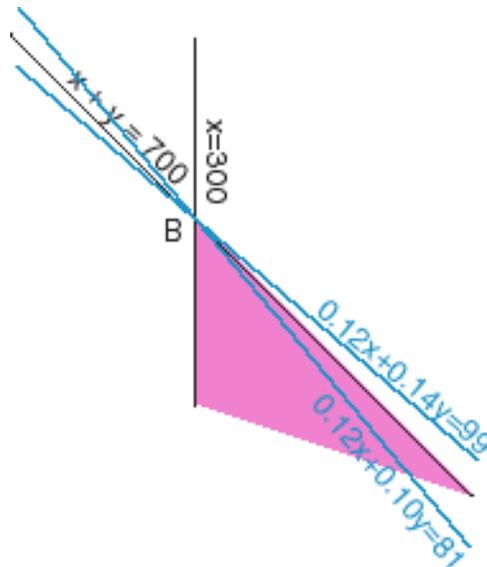
This can be inferred from the Sensitivity report given by Excel, too. The allowable decrease in the coefficient of investments in the objective function is .02. We shifted it .04, and thus we changed the solution.

III.E. [BHM, Exercise 1.14] *A strategic planner for an airline that flies to four different cities from its Boston base owns 10 large jets (B707s), 15 propeller-driven planes (Electras), and two small jets (DC9s). Assuming constant flying conditions and passenger usage, the following data is available.*

	City	Round Trip Cost	Round Trip Revenue	Average flying time (hours
B707	A	$6,000	$5,000	1
	B	7,000	7,000	2
	C	8,000	10,000	5
	D	10,000	18,000	10
Electra	A	1,000	3,000	2
	B	2,000	4,000	4
	C	4,000	6,000	8
	D	—	—	20
DC9	A	2,000	4,000	1
	B	3,500	5,500	2
	C	6,000	8,000	6
	D	10,000	14,000	12

Formulate constraints to take into account the following:

  (a) City D must be served twice daily; cities A, B, and C must be served four times daily;
  (b) Limitation on number of planes available, assuming that each plan can fly at most 18 hours per day

Formulate objective functions for:

  (a) Cost minimization;
  (b) Profit maximization
  (c) Fleet flying-time minimization

Indicate when a continuous linear-programming formulation is acceptable, and when an integer-programming formulation is required.

Challenge: Compute, compare and contrast solutions which optimize these objectives.

We'll do this all on Excel. We make a decision variable for the number of flights by each plane type to each city.

	A	B	C	D	E	F
1			Valujet			
2			*We get there 90% of the time!*			
3			Boston Routing Plan			
4						
5						
6	**No. Planes**			Max. Hours/plane/day		
7	B707	10		18		
8	Electra	15				
9	DC9	2				
10						
11	**No. Flights**		City			
12	Plane	A	B	C	D	Flights by Plane Type
13	B707	13	2	4	3	22
14	Electra	3	2	7	4	16
15	DC9	1	6	7	5	19
16	Flights by City	17	10	18	12	
17	Constraints (min)	4	4	4	2	
18						
19	**Trip Costs**		City			
20	Plane	A	B	C	D	Costs by Plane
21	B707	6000	1000	8000	10000	142000
22	Electra	1000	2000	4000		33000
23	DC9	2000	3500	6000	1000	60000
24	Costs by City	83000	27000	102000	35000	235000

	A	B	C	D	E	F
26	**Trip Revenue**		City			
27	Plane	A	B	C	D	Revenue by Plane
28	B707	5000	1000	10000	18000	161000
29	Electra	3000	4000	6000		71000
30	DC9	4000	5500	8000	14000	137000
31	Revenue by City	78000	43000	138000	124000	369000
32						
33	**Trip Flight Time**		City			
34	Plane	A	B	C	D	Flight Time by Plane	Available
35	B707	1	2	5	10	67	180
36	Electra	2	4	8	20	126	270
37	DC9	1	2	6	12	77	36
38	Flight Time by City	20	24	118	170	270	
39							
40	Profit	$134,000.00					

All formulas are written with the various sumproducts. The constraints can be entered as $B$16:$E$16 >= $B$17:$E$17 and $F$35:$F$37<=$G$35:$G$37. Oh, and since for some reasons flying Electras to city D are not allowed, we require $E$14=0. Now we just run solver three times.

(a) First, we ask to minimize the total costs (F24). We get an answer saying we should send 4 Electras to city A, 4 B707s to city B, 4 Electras to city C, and 2 DC9s to city D. This sets costs at $26,000.

(b) We first solve this with a continuous model and interpret the answer. Solver tells us to send 135 Electras and 36 DC9s to city A each day, 4 B707s to city B, 4 B707s to

city C, and 15.2 B707s to city D. Many times, we can interpret the solution as a rate. So we could try sending 16 B707s to D on one day, then 15 on the next four days. In the average, we would be right on 15.2 per day. This gives a profit of $471,600. However, the planes are strictly kept (probably by law) to fly no longer than 18 hours on *any* day, and taking 16 flights to D would violate that (you should check that this constraint is binding). So we have to use an integer programming method. Once we constrain the values of the decision variables to be integers, we come up with the solution: send 127 Electras and 36 DC9s to A, 4 Electras to B, 4 B707s to C, and 16 B707s to D. This results in a profit of $470,000.

(c) Solving to minimize the total flight time per day (F38) gives the answer that we should only fly with the B707s. Take 4 flights per day to cities A, B, C, and 2 flights per day to city D.