

QR26: Choice and Chance

Solutions to Unit I Exercises

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22 February 2000

I.A Problem 2 from the UMAP Module “Measurement Scales” by Jo Anne B. Growney:

Karen, Leon, and Murray just learned of their scores on the math test they took last Friday. Karen’s score was a 95 out 100 possible points. Leon’s was 70, Murray’s was 35. Which of the following are reasonable conclusions to draw from this information?

- (a) *Leon knows twice as much mathematics as Murray does.*
- (b) *Karen got more test answers correct than Leon did.*
- (c) *The amount of mathematics that Leon knows is closer to the amount that Karen knows than it is to the amount Murray knows.*

Think about: Just what does a mathematics test (or any other test for that matter) measure? How meaningful are tests as measures of what an individual knows?

A test measures whether a student knows the answers to various questions, precisely the ones on the test. Whether the results are representative of the student’s mastery of the subject depends on the questions. Major test-writing companies (e.g., ETS) strive for tests which accurately reflect knowledge and aptitude.

However, unless we know what’s on the test, we can’t conclude much. All conclusions are based on the test, and therefore can only concern the test. This rules out (a) and (c). (b) is a more reasonable conclusion to draw. There are situations (think of questions with different point values) where the person with a higher score got fewer questions right.

I.B *Ibid.*, Problem 8:

Average per capita income in Florida was \$3700 in 1970 and \$6700 in 1977. Is it valid to conclude that the average Floridian was better off in 1977? Discuss briefly.

The answer is *no*, because the two dollar amounts are not actually the same units! The figure of \$6700 is in 1977 dollars, and the \$3700 figure is in 1970 dollars. They are not measured on the same scale. Therefore, a difference in values is meaningless.

There are other problems with such a conclusion. The average per capita income would also be affected by an immigration of rich people and/or an emigration of poor people.

A better way to measure whether the “average” Floridian is better off is to compare income person by person between 1970 and 1977, adjusted for inflation, and take *that* average. Notice this is an average of differences rather than a difference of averages!

I.C *Ibid.*, Problem 3:

(a) *When grades (such as A, B, C, D, F) are assigned to a student at the end of a course, what type of measurement scale is involved? Is there a difference if numerical values (such as 4, 3, 2, 1, 0) are used?*

(b) *Is the grade point average a meaningful quantity? Discuss*

(a) Imagine switching grades from letters to numbers: $A \leftrightarrow 4$, $B \leftrightarrow 3$, and so on. Are you adding any information about the relative values of grades? Of course not. Whether letters or numbers are used, the scale is an ordinal one. The use of numbers does not make it an interval scale, and the use of a zero (instead of F) does not make it a ratio scale. Each grade refers to a range of scores in a given class, and comparisons of differences between grades are meaningless. Nor does a grade of F or 0 mean that the student did no work.

Here is another justification why letter grades cannot be an interval scale. Suppose that on a test, A represents a score between 90% and 100%, B the range 80%–89%, and C the range 70%–79%. You expect that the difference between an A and a B would be the same as the difference between a B and a C. However, try this with scores of 90, 89, and 70. The difference between this particular A and this particular B is 1%, but the difference between the B and the C is 19%! Imagine if you tried to assign the 4, 3, 2, 1 scale to these letter grades and interpret the “averages.”

(b) If the numbers assigned to grades in a student’s courses are not an interval scale, then the sums of them are not a meaningful quantity. Hence the averages are not meaningful, either, inasmuch as they do not provide a basis for comparison of students’ work.

I.D *Serving on a national fellowship selection committee, you notice that the systems for scaling letter grades to compute class averages is different at Harvard than from other colleges. If student record x ranks higher than y at Harvard, can you conclude that the same grade record would also rank*

higher than y at other colleges? Does it make sense to say Record x is 10% better than Record y ? Record x is twice as strong as Record y ? Does it make a difference if the two averages are computed on the same or different scales? By what system would you rank students based on their grades and with what reservations?

We know that there can be ambiguities from system to system; we saw an example in class. Thus, comparisons cannot be made from scale to scale. This is why selection committees (like for graduate schools) ask for the transcript and compute their own averages. Still, one could try to correct further by adjusting each school's grades for "grade inflation." A way to do this might be to compare the student's grades with the average grades given by that school. Even once a scale is fixed, however, it is still an ordinal one and not an interval one. So no quantitative statements can be made about differences in grades.

- I.E *Starting from the assumption that both are interval scales, derive formulae for switching between Fahrenheit and Centigrade temperatures by imposing what you know about the freezing and boiling points of water in each system. Show that, in contrast to the previous example, the question of whether the average temperature (as measured each day at noon, say) in City X is bigger than in City Y over a given period does not depend on which scale you use in both.*

To "derive" means to do something like the following, and not just write down the answer. Since both are interval scales, the differences in equal heats must be proportional. We know that water freezes at $32^\circ\text{F} = 0^\circ\text{C}$ and boils at $100^\circ\text{F} = 212^\circ\text{C}$. Therefore, it is safe to say that a change in heat of $100 - 0 = 100^\circ\text{C}$ is the same as a change in $212 - 32 = 180^\circ\text{F}$.

Dividing, we see that a difference of 1°C is the same as a difference in $\frac{180}{100} = \frac{9}{5} = 1.8^\circ\text{F}$. This means for each degree Celsius more than zero, we must add 1.8 degrees Fahrenheit to 32. Thus:

$$F = \frac{9}{5}C + 32$$

We have its counterpart:

$$C = \frac{5}{9}(F - 32).$$

Since the conversion formula is in each case an increasing linear function, we can quantitatively say things like "the average temperature in City X is greater than that in City Y." To see this, let F_1, \dots, F_n be the temperatures (measured on the Fahrenheit scale) in City X at certain times, and F'_1, \dots, F'_n the temperatures measured in Fahrenheit in City Y at the exact same times. Then to say that the average Fahrenheit temperature in City

X is higher than that in City Y means that

$$\bar{F} \frac{1}{n} \sum_{i=1}^n F_i > \frac{1}{n} \sum_{i=1}^n F'_i = \bar{F}'.$$

Now let $C_1, \dots, C_n, C'_1, \dots, C'_n$ be the same temperature conditions measured in cities X and Y as before, but this time in Celsius. We know that

$$C_i = \frac{5}{9}(F_i - 32)$$
$$C'_i = \frac{5}{9}(F'_i - 32).$$

Then the average temperature in Fahrenheit, converted to Celsius, is the same as the average of the converted temperatures:

$$\begin{aligned} \bar{C} &= \frac{1}{n} \sum_{i=1}^n C_i \\ &= \frac{1}{n} \sum_{i=1}^n \frac{5}{9}(F_i - 32) \\ &= \frac{5}{9} \left(\left(\frac{1}{n} \sum_{i=1}^n F_i \right) - 32 \right) \\ &= \frac{5}{9}(\bar{F} - 32). \end{aligned}$$

This means that $\bar{F} > \bar{F}'$ if and only if $\bar{C} > \bar{C}'$. The point is that it doesn't matter in which order you apply the averaging and converting transformations.

For fun, visit the URL <http://www.strug.com.au/~mbug/old.html>.

I.F *Do at least one problem from the Kreps chapters in the sourcebook.*

The problem goes on with the apology “This material is pretty technical. Just make sure you understand what a binary relation is, what properties it must have to be called a preference relation, and when such a relation determines a choice function.” This is Aristotelian teaching (cf. *Posterior Analytics*): in order to understand what something *is*, one must also understand what it *is not*. Most students who did a problem tried Problem 1 of Chapter 2, which was not the easiest one, by the way. As a study aide, we include answers to all of the Chapter 2 problems.

(1) *For each of the five examples on page 7, show that the binary relation has precisely the list of properties that are ascribed to it on page 8.*

(a) *Let $X = \{1, 2, 3\}$ and $B = \{(1, 1), (1, 2), (1, 3), (2, 3), (3, 1)\}$.*

We can draw this relation in a table. I also included a few equals signs to demonstrate weak connectivity.

	1	2	3
1	B	B	B
2		=	B
3	B		=

Now we demonstrate why B has none of the other listed properties. It is not reflexive because $2\tilde{B}2$. It is not irreflexive, either, because $1B1$. It is not symmetric, because $1B2$ but $2\tilde{B}1$. Nor is it asymmetric, because $1B3$ and $3B1$. It's not transitive because $2B3$ and $3B1$, but $2\tilde{B}1$. It's not negatively transitive because $3\tilde{B}2$ and $2\tilde{B}1$, but $3B1$. It's not complete because $2\tilde{B}2$ (note completeness implies reflexivity by setting one side equal to the other). And it's not acyclic because $1B2$, $2B3$, and $3B1$. It does not like it in a car; it does not like it in a bar. It does not like green eggs and ham. It does not like it, Sam-I-Am!

- (b) *Let X be the set of all people in the world and let B be the relation "shares at least one given name with."*

B is reflexive—everybody shares all of their given names with themselves. It's symmetric—if I share a given name with you, you share the same given name with me. Notice this relation is not transitive. Consider the people named Joe Bob Briggs, Billy Joe Tolliver, and Billy Bob Thornton. Oh, wait, that triple is transitive. Throw in Frankie Lee, and Judas Priest, and me and Bobby McGee, and now you see that Joe Bob Briggs B 's Bobby McGee and Billy Joe Tolliver, but Bobby McGee \tilde{B} 's Billy Joe Tolliver.

- (c) *Let $X = \mathbb{R}$ and let B be the relation \geq .*

This relation is reflexive, antisymmetric, transitive, negatively transitive, complete, and weakly connected. Any proof of this that isn't completely tautological would define \geq to mean $x \geq y$ if $x - y \geq 0$. You would then have to use the axiom that any real number is either positive, negative or zero. And this, believe it or not, all follows from the fact that 1 must be positive. It's not obvious why that reduction is possible, but can you prove 1 is positive?

- (d) *Let $X = \mathbb{R}$ and let B be the relation: xBy if $|x - y| > 1$.*

B is irreflexive since $|x - x| = 0 < 1$ (see the previous discussion!), so $x\tilde{B}x$ for all $x \in \mathbb{R}$. It's symmetric because $|x - y| = |y - x|$.

- (e) *Let $X = \mathbb{R}$ and let B be the relation: xBy if $x - y$ is an integer multiple of 2.*

There is a real name for this relation. It's called "congruence modulo 2" and is written $x \equiv y \pmod{2}$. It is indeed reflexive, because $x - x = 0 = 2 \cdot 0$, so $x \equiv x \pmod{2}$. It's symmetric, since if $x \equiv y \pmod{2}$, there is an integer k such that $x - y = 2k$, and therefore $y - x = 2(-k)$. $-k$ is a perfectly good integer, too, so $y \equiv x \pmod{2}$. Now suppose $x \equiv y \pmod{2}$ and $y \equiv z \pmod{2}$. Then there exists a k as before and an integer ℓ such that $y - z = 2\ell$.

Then

$$x - z = (x - y) + (y - z) = 2k + 2\ell = 2(k + \ell),$$

so $x \equiv z \pmod{2}$.

- (2) A binary relation E that is reflexive, symmetric, and transitive is called an equivalence relation. (For example, if \succ is a preference relation, then \sim is an equivalence relation; cf. Proposition (2.4)(c).) Here is an easy proof that if E is symmetric and transitive, then it is automatically reflexive (thus reflexive could be deleted from the list of properties): Fix $x \in X$ and take some y such that xEy . Then yEx by symmetry, and hence xEx by transitivity. Unfortunately, this easy proof is spurious. Why?

“Spurious” means illegitimate or false. The trap we fell into was assuming that a $y \in X$ such that xEy exists at all. Consider this statement: “While some male popes were sinners, all female popes were saints.” This is logically correct, but no conclusion (for instance, “women make better popes than men”) can be drawn from it.

To prove that the assumption of reflexivity is not redundant, we must exhibit a relation which is symmetric and transitive, but not reflexive. Consider the relation E on a two-point set $\{x, y\}$ given by

	x	y
x		
y		E

- (3) Prove Proposition (2.5): Given a binary relation \succeq' on a set X , define two new binary relations \succ' and \sim' from \succeq' by

$$x \succ' y \text{ iff } y \not\succeq' x, \text{ and } x \sim' y \text{ iff } x \succeq' y \text{ and } y \succeq' x.$$

Then if \succeq' is complete and transitive, \succ' will be a preference relation. Moreover, if we start with a binary relation \succeq' , define \succ' and \sim' as above from \succeq' , and then define \succeq and \sim from \succ' by

$$x \succeq y \text{ iff } y \not\succeq' x, \text{ and } x \sim y \text{ iff } y \not\succeq' x \text{ and } x \not\succeq' y,$$

then \succeq' and \succeq will agree, as will \sim' and \sim .

We’ve used “iff” as shorthand for “if and only if” (this is standard in mathematics), which means that two statements are logically equivalent: one is true exactly when the other is true. One can also use the symbol \iff , which is built of the symbol \implies (“implies”) and \impliedby (“is implied by”).

Let \succeq' be given, and define \succ' and \sim' as above. We know that \succeq' is complete and transitive, and we want to show that \succ' is a preference relation, meaning it’s asymmetric and negatively transitive. Here is the proof of asymmetry: Give me any x and y in X such that $x \succ' y$. This

means precisely that $y \not\prec' x$. By completeness of \succeq' , then, we must have that $x \succeq' y$. This says precisely that $y \not\prec' x$. Since this is true for arbitrary x and y , \succ' must be asymmetric.

Now to show the negative transitivity: Give me x , y , and z in X such that $x \not\prec' y$ and $y \not\prec' z$. I must show that $x \succ' z$. Again, by definition of \succ' and completeness of \succeq' , I know that $y \succ' x$ and $z \succ' y$. Hence $z \succ' x$, in other words $x \not\prec' z$. Since this is true for any three elements of X , \succ' is negatively transitive.

Now define \succeq and \sim as above. We need to show that these relations are equivalent to \succeq' and \sim' , respectively. This means that for any x and y ,

$$x \succeq y \iff x \succeq' y,$$

and the same for \sim . Well, give me x and y . Then

$$\begin{aligned} x \succeq y &\iff y \not\prec' x && \text{(Definition of } \succeq) \\ &\iff x \succeq' y && \text{(Definition of } \succ') \end{aligned}$$

And

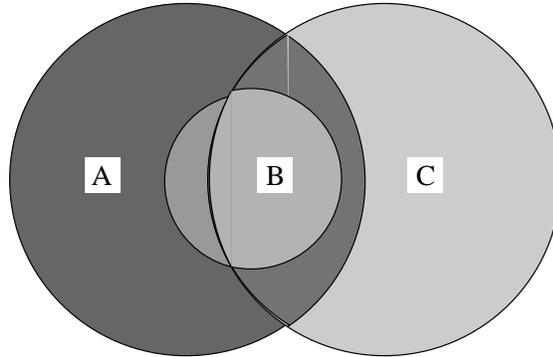
$$\begin{aligned} x \sim y &\iff y \not\prec' x \text{ and } x \not\prec' y && \text{(Definition of } \sim) \\ &\iff x \succeq' y \text{ and } y \succeq' x && \text{(Definition of } \succ') \\ &\iff x \sim' y && \text{(Definition of } \sim') \end{aligned}$$

- (4) *Show that the properties (for a binary relation) of asymmetry and negative transitivity are independent.*

To say that properties p and q are *independent* is to say that p does not imply q and q does not imply p . So we need to exhibit two relations, one of which is asymmetric but not negatively transitive and one which is negatively transitive but not asymmetric.

The second is easier: Consider the relation \geq on \mathbb{R} . Then the negative of \geq is $<$, and $<$ is transitive, so \geq is negatively transitive. But \geq is reflexive ($x \geq x$ for all $x \in \mathbb{R}$), and so it can't be asymmetric (because then $x \not\geq x$).

For the first, consider the relation \subset on subsets of a set X , where in this paragraph we say $A \subseteq B$ if all elements of A are elements of B , and $A \subset B$ if $A \subseteq B$ but $A \neq B$. Many times equality is included (more precisely, not excluded) from the definition of \subset , but not right now. We claim \subset is asymmetric but not negatively transitive. Give me sets A and B such that $A \subset B$. This means all elements of A are also elements of B , but at least one element of B is not an element of A . Therefore, $B \not\subset A$. Hence \subset is asymmetric. But try the example where X is the set $\{1, 2, 3, 4, 5\}$, and $A = \{2, 3, 4\}$, $B = \{3, 4, 5\}$, and $C = \{1, 2, 3, 4\}$. Then $A \not\subset B$ (because 2 is not in B), $B \not\subset C$ (because 5 is not in C), yet $A \subset C$. Therefore, \subset is not negatively transitive. Here is a picture of the same proof:



- (5) Prove Proposition (2.8): For a binary relation \succ (on a finite set X), $c(\cdot, \succ)$ is a choice function if and only if \succ is acyclic. In what sense is it important here that X is finite?

Recall that

$$c(A, \succ) = \{x \in A \mid \forall y \in A, y \not\succeq x\}.$$

The “ \forall ” symbol means “for every.” The implication “ \Leftarrow ” (that is, \succ is acyclic implies that $c(\cdot, \succ)$ is a choice function) was already proved in Proposition (2.7). We need to prove the converse.

So let $c(\cdot, \succ)$ be a choice function; we need to show \succ is acyclic. Given elements x_1, x_2, \dots, x_{n+1} such that $x_1 \succ x_2, x_2 \succ x_3, \dots, x_n \succ x_{n+1}$, we must show that $x_1 \not\succeq x_{n+1}$. Well, suppose it were. Then $x_n \succ x_1$. What is the set $c(\{x_1, \dots, x_n\}, \succ)$? It can't contain x_n , because that's dominated by x_{n-1} , because that's dominated by x_{n-2} . And so on \dots . It can't contain x_2 , because that's dominated by x_1 , and it can't contain x_1 , because that's dominated by x_n ! Hence $c(\{x_1, \dots, x_n\}, \succ) = \emptyset$, and $c(\cdot, \succ)$ can't be a choice function.

This part of the proof does not use the finiteness of X . The initial implication, Proposition (2.7), *does* use the finiteness of X by noting that any subset $A \subset X$ is finite. Finiteness of X is in fact necessary (that is, it was not a needless assumption), because the proposition is not true for infinite X . For instance, take $X = \mathbb{R}$, let \succ be the relation $>$, and consider $c(\mathbb{Z}, >)$. (\mathbb{Z} is the set of integers.) Since there is no “largest” integer, $c(\mathbb{Z}, >)$ must be empty. Hence $c(\cdot, >)$ is not a choice function.

- (6) Prove that for any binary relation \succ , $c(\cdot, \succ)$ defined as in the display on page 12 satisfies Sen's α . (Is it important here that X is finite? In what sense is it important that \succ is acyclic, even though you won't told to assume acyclicity?)

Give me subsets $B \subset A \subset X$, and $x \in B$. Suppose $x \in c(A, \succ)$. We need to show $x \in c(B)$, i.e., that x is a Pakistani champion. Well, by definition

$$x \in c(A, \succ) \implies \forall y \in A, y \not\succeq x$$

In particular,

$$\begin{aligned} &\implies \forall y \in B, y \neq x \\ &\implies x \in c(B, \succ). \end{aligned}$$

So Sen's Axiom α is satisfied. Again, to assume the sets $c(A, \succ)$ and $c(B, \succ)$ are nonempty, we must assume \succ is acyclic.

- (7) *Give an example of a finite set X and an acyclic binary relation where $c(\cdot, \succ)$ does not satisfy Sen's property β .*

Think small! Since $c(\cdot, \succ)$ satisfies Houthakker's axiom (and therefore Sen's axiom β) when \succ is a preference relation, any counterexample we propose had better not be both asymmetric and negatively transitive. Let's destroy negative transitivity, and try a relation defined on a three-point set $X = \{x, y, z\}$ by

	x	y	z
x			
y			\succ
z	\succ		

Then $z \not\succeq y$, and $y \not\succeq x$, but $z \succ x$, so \succ is not negatively transitive. Notice that it is asymmetric, though. Anyway, let $A = \{x, y\}$ and $B = \{x, y, z\}$. Then $c(B, \succ) = \{y\}$ and $c(A, \succ) = \{x, y\}$. Hence y is a world champion and a Pakistani champion, but x is a Pakistani champion that is not a world champion. Thus Sen's β is not satisfied.

- (8) *In and around Propositions (2.13) and (2.14), I seem to get very confused about whether negative transitivity is all I need to prove Houthakker's axiom, and thus asymmetry. Unconfuse me. Have I implicitly used asymmetry in the proof of (2.13), and if so, where?*

Suppose \succ is not asymmetric. Unraveling the definition, this means there exists a pair of elements x and y in X such that $x \succ y$ and $y \succ x$. Then by the same argument that proved Proposition (2.8), $c(\{x, y\}, \succ) = \emptyset$, which is illegal. So asymmetry is necessary to even start the problem.

- (9) *Give an example of a finite set X and a choice function c on $P(X)$ that satisfies Sen's α but such that there is no binary relation \succ such that $c(\cdot, \succ) = c$.*

After playing with a three-element set, we pull the rabbit out of the hat. Define a choice function by first setting

$$\begin{aligned} c(\{x, y, z\}) &= \{y\}; \\ c(\{x, y\}) &= \{x, y\}. \end{aligned}$$

Then Sen's axiom β is not satisfied by this choice function. Notice that for Sen's α to hold, we must have

$$c(\{y, z\}) = \{y\}.$$

Finally, the last choice is arbitrary:

$$c(\{x, z\}) = \{x, z\}.$$

(On the three singleton sets we must have $c(\{x\}) = \{x\}$, et cetera.)

- I.G *You are decorating a room for an indecisive grandparent, who will not tell you what color paint to use, but will give you a strict and consistent preference between any pair of colors. The only choices are Lilac, Aqua, Brown, Muscroom, White, Green, and Pink. Using first letters as abbreviations, you have asked seven questions and learned that your grandparent holds:*

$$M > P, M > G, P > W, A > M, G > W, B > L, \text{ and } L > M.$$

What is the next and last question you ask in order to determine what color to paint the room? In all, you have asked eight questions. What is the least number you could have asked, and is there a strategy for questioning that ensures you only ask the least number possible? Give reasons for your answer.

Each time your grandparent give you a choice, you can “cross out” the non-preferred color, because it can’t possibly be optimal. You are left with only A and B un-crossed-out. So ask whether your grandparent prefers Aqua to Brown.

A simpler algorithm is called “King of the Mountain.” Pick any two colors, and ask your grandparent which they prefer. Take the “winner” and compare it with any other. Proceed in this process, taking the more preferred element. After six questions, you will have eliminated six colors, leaving the one winner.

You could also do a “tournament” by splitting up colors into pairs and comparing them against each other. Any leftover singleton is given a “bye.” After the third round, the champion is the winner. There are still only six games.

Why is six the minimum? Each comparison eliminates at most one element (you may have eliminated it before if you’re not being extremely efficient). Hence you need at least six comparisons to eliminate the six non-winners.

- I.H *You are a judge in an amateur pie baking contest and must pick a winner from among four entries labeled x , y , z , and w . You rank each according to seven attributes, say Appearance, Calories, Flavor, Novelty, Smell, Texture, and Vitamins. The highest ranked pie in each category gets 4 points, the next highest 3, etc. The one with the highest number of points when added all together wins. This method is sometimes called a Borda count.*

	A	C	F	N	S	T	V	Total
X	4	1	2	4	2	2	4	18
Y	3	4	1	3	4	1	3	19
Z	2	3	4	2	3	4	2	20
W	1	2	3	1	2	3	1	13

- i. After z is declared the winner, it is discovered that w should not have been disqualified because it was baked by a professional. This should not matter, it seems, since z was found better than w in every category. But would z have won if there were only three choices to begin with so that the top ranked got 3 points and so on?

We can re-do the Borda count like this:

	A	C	F	N	S	T	V	Total
X	3	1	2	3	1	2	3	15
Y	2	3	1	2	3	1	2	14
Z	1	2	3	1	2	3	1	13

Now x is the winner!

- ii. Referring to Chapter 2 [typo] of Kreps, what would Sen say this means about the choice function c that selects the Borda winner out of each subset of $\{x, y, z, w\}$? Are there preferences \succ on the set such that the Borda choice function is of the form $c(\cdot, \succ)$?

Since $c(\{x, y, z, w\}) = z$ and $c(\{x, y, z\}) = x$, we see that c does not satisfy Sen's axiom α , a/k/a the independence of irrelevant alternatives. Thus there cannot be a preference relation that reproduces c .

- iii. What do you think is fair to do about awarding the pie-baking prize? Do you think it becomes easier or harder to pick a winner fairly if the judge is allowed to rate rather than rank the entries according to each attribute?

If each pie is given a score on an interval scale, then adding up (or forming an average) of the ratings does make sense (think figure skating). Disqualifying a single contestant does not change the scores of other contestants.

- iv. Mathematical Challenge: When doing a Borda count like this with n alternatives, each column must add up to $1 + 2 + 3 + \dots + n$. Find and justify a formula for this sum. Look up the legend about how the mathematician Gauss figured out the answer as a schoolboy.

"Justify" means prove, of course. No credit was given for writing down the formula. Here is the formula: Let the sum of the first n numbers be N . Then

$$N = 1 + 2 + \dots + (n - 1) + n.$$

On the other hand,

$$N = n + (n - 1) + \dots + 2 + 1.$$

Add these two up!

$$\begin{aligned} 2N &= \underbrace{(n+1) + (n+1) + \cdots + (n+1)}_{n \text{ summands}} \\ &= n(n+1). \end{aligned}$$

Hence

$$N = \frac{n(n+1)}{2}.$$

Here is the story from *Men of Mathematics*, ©1937 by E.T. Bell:

...in his tenth year, Gauss was admitted to the class in arithmetic. As it was the beginning class, none of the boys had ever heard of an arithmetical progression. It was easy then for the heroic Büttner [his teacher] to give out a long problem in addition whose answer he could find by a formula in a few seconds. The problem was of the following sort: $81297 + 81495 + \dots + 100899$, where the step from one number to the next is the same all along (here 198), and a given number of terms (here 100) are to be added.

It was the custom of the school for the boy who first got the answer to lay his slate on the table; the next laid his slate on top of the first and so on. Büttner had barely finished stating the problem when Gauss flung his slate on the table: "There it lies," he said—"Ligget se' " in his peasant dialect. Then, for the ensuing hour, while the other boys toiled, he sat with his hands folded, favored now and then by a sarcastic glance from Büttner, who imagined the youngest pupil in the class was just another blockhead. At the end of the period, Büttner looked over the slates. On Gauss's slate there appeared but a single number. To the end of his days Gauss loved to tell how the one number he had written was the correct answer and how all the others were wrong. Gauss had not been shown the trick for doing such problems rapidly. It is very ordinary once it is known, but for a boy of ten to find it instantaneously by himself is not so ordinary.

- v. Research Challenge: *Find out to what extent the Borda count describes how winners are picked in Consumer Reports, in Cambridge elections, in the Heisman Trophy competition, et cetera.*

Most did not bother to do this, but those who did were awarded extra credit. The Heisman Trophy is chosen by a group of 785 writers, who rank three finalists. A rank of first is given three points, second two points, and third one point. Is it possible that eliminating a non-winning contestant could change the winner? (See if you can find this phenomenon with a smaller number of writers).

The Cambridge City Council elections are strange indeed. Nine councilors must be elected, and there is a quota of votes necessary to be elected. Voters rank their choices. On the first count, each ballot with a candidate in the number one slot causes a vote to be awarded, up to the point where the candidate meets the quota. From that point on, the ballot counts as a vote to a lower-ranked candidate. After all the ballots are counted once, anybody with fewer than 50 votes is eliminated, and their votes are redistributed to the other candidates, continuing to go by the rule that a ballot will count as a vote to the highest-ranked candidate not already elected. After the second go-around, the candidate with the lowest vote count is eliminated, and his/her votes are redistributed. The process continues until the nine winners remain.

So it seems like who your vote goes to depends on the order the votes are counted, doesn't it? Can you prove or disprove that the elimination of a non-winning candidate after the polls close but before the votes are counted changes the outcome?

Visit the URL <http://www.ci.cambridge.ma.us/~Election/prop-voting.html> for more information.

I.I Optional Mathematical Exercise: *To talk about the best alternatives in a [finite] set, we introduce the function \mathbf{B} that operates on sets to produce the best of that set. We allow for ties so for a given set V , the set $\mathbf{B}(V)$ may consist of more than one element.*

We assume B has the following properties:

1. *For any nonempty set V , $\mathbf{B}(V)$ is not empty. This implies all elements are [weakly] comparable.*
2. *$\mathbf{B}(V) \subset V$ for any set V .*

Conditions 1 and 2 imply that for any element x ,

$$\mathbf{B}(\{x\}) = \{x\}.$$

3. ***Independence of Irrelevant Alternative (IIA) Condition:*** *For any sets V and W ,*

$$\mathbf{B}(V \cup W) = \mathbf{B}[\mathbf{B}(V) \cup \mathbf{B}(W)]$$

Questions:

- a. *Interpret the meaning of Condition 3.*
- b. *Show that if element x belongs to V but not to $\mathbf{B}(V)$, then x does not belong to $\mathbf{B}(V \cup \{y\})$, i.e., you should not be able to turn a non-best element in a set into a best element by enlarging the set.*
- c. *Show that if x belongs to V but not to $\mathbf{B}(V)$, then $\mathbf{B}(V \setminus \{x\}) = \mathbf{B}(V)$, i.e., deleting a non-best element from V does not change what's best in the depleted set.*

Let's do these before (d). We know that (a) is also known as Sen's axiom α , and can be thought of like this as well: If you're choosing from items on a menu, then the best on pages A and B is the better of the best of page A and the best of page B .

For (b), we have

$$\mathbf{B}(V \cup \{y\}) = \mathbf{B}[\mathbf{B}(V) \cup \{y\}] \subset \mathbf{B}(V) \cup \{y\}.$$

Since x is neither in the set $\mathbf{B}(V)$ nor in $\{y\}$ (the assumption $x \neq y$ has been implicit), then $x \notin \mathbf{B}(V \cup \{y\})$.

(c) is a little harder. First, note $\mathbf{B}[\mathbf{B}(V)] = \mathbf{B}(V)$. This is because of IIA and some set theory:

$$\begin{aligned} \mathbf{B}(V) &= \mathbf{B}(V \cup \mathbf{B}(V)) \\ &= \mathbf{B}[\mathbf{B}(V) \cup \mathbf{B}(\mathbf{B}(V))] \\ &= \mathbf{B}[\mathbf{B}(V)]. \end{aligned}$$

You should be able to justify each equals sign. Now notice that

$$\begin{aligned} \mathbf{B}(V) &= \mathbf{B}((V \setminus \{x\}) \cup \{x\}) \\ &= \mathbf{B}[\mathbf{B}(V \setminus \{x\}) \cup \{x\}] \\ &\subset \mathbf{B}(V \setminus \{x\}) \cup \{x\}. \end{aligned}$$

Since x is not in $\mathbf{B}(V)$ or $\mathbf{B}(V \setminus x)$, we must have that $\mathbf{B}(V) \subset \mathbf{B}(V \setminus \{x\})$.

Finally, notice that

$$\begin{aligned} \mathbf{B}(V) &= \mathbf{B}((V \setminus \{x\}) \cup V) \\ &= \mathbf{B}[\mathbf{B}(V \setminus \{x\}) \cup \mathbf{B}(V)] \\ &= \mathbf{B}[\mathbf{B}(V \setminus \{x\})] \\ &= \mathbf{B}(V \setminus \{x\}). \end{aligned}$$

The Preference Relations P and I associated with \mathbf{B} .

Definition: For any two-element set $\{x, y\}$, set

$$xPy \iff \mathbf{B}\{x, y\} = \{x\}$$

and

$$xIy \iff \mathbf{B}\{x, y\} = \{x, y\}$$

d. Prove the following theorem: If B satisfies conditions 1, 2, 3, then its associate preference relation P is transitive.

Suppose not. Then there exist three different elements x , y , and z such that xPy , yPz , and xPz . This means that $\mathbf{B}\{x, y\} = \{x\}$, $\mathbf{B}\{y, z\} = \{y\}$, but $\mathbf{B}\{x, z\} \neq x$.

So we have two cases for $\mathbf{B}\{x, z\}$. The first possibility is that $\mathbf{B}\{x, z\} = \{z\}$, or zPx . Let's try to find the best of the set $\{x, y, z\}$.

$$\begin{aligned}\mathbf{B}\{x, y, z\} &= \mathbf{B}(\{x, y\} \cup \{z\}) \\ &= \mathbf{B}(\mathbf{B}\{x, y\} \cup \{z\}) \\ &= \mathbf{B}\{x, z\} = \{z\}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathbf{B}\{x, y, z\} &= \mathbf{B}(\{x, z\} \cup \{y\}) \\ &= \mathbf{B}(\mathbf{B}\{x, z\} \cup \{y\}) \\ &= \mathbf{B}\{z, y\} = \{y\}.\end{aligned}$$

This is a contradiction as long as $y \neq z$.

The other case is that $\mathbf{B}\{x, z\} = \{x, z\}$. So we try to find $\mathbf{B}\{x, y, z\}$ again.

$$\begin{aligned}\mathbf{B}\{x, y, z\} &= \mathbf{B}[\{x, y\} \cup \{z\}] \\ &= \mathbf{B}\{x, z\} = \{x, z\}\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathbf{B}\{x, y, z\} &= \mathbf{B}[\{y, z\} \cup \{x\}] \\ &= \mathbf{B}\{y, x\} = \{x\}.\end{aligned}$$

This is a contradiction since $x \neq z$. So we must have made a mistake, and that was supposing that P was not transitive.