UNIT 546

umap

MODULES AND MONOGRAPHS IN UNDERGRADUATE
MATHEMATICS AND ITS APPLICATIONS PROJECT

MEASUREMENT SCALES

by Jo Anne S. Growney

ANDERSON : (First choice)
BUSH : (Second choice)
CONNALLY : (Fourth and last choice)
REAGAN : (Third choice)

AMERICAN REVOLUTION : 1775
CIVIL WAR : 1861
SPANISH AMERICAN WAR : 1898
WAR OF 1812 : 1812

APPLICATIONS OF ARITHMETIC TO
PHYSICAL AND SOCIAL PROBLEMS

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications that may be used to supplement existing courses and from which complete courses may eventually be built.

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MEASUREMENT SCALES

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Intermodular Description Sheet: UMAP Unit 546

Title: MEASUREMENT SCALES

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Classification: APPL ARITH/PHYSICAL & SOCIAL PROBLEMS

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Target Audience: Students in a general education survey mathematics course.

Abstract: This unit introduces four types of measurement scales -- nominal, ordinal, interval, and ratio -- and considers their uses and limitations. The question, "When may two measurements be compared meaningfully?" is posed and answered. The concept of "utility" is introduced. Students learn to interpret different measurement scales, see a variety of uses of measurement scales, and learn to invent new scales to solve problems.

Prerequisites: Any student with a knowledge of arithmetic is prepared for the mathematical content of this module.

Suggestions for Effective Use of this Module:

This module has been class tested by the author as one of the beginning units in a general education mathematics course for college freshmen that emphasizes problem solving. Effective class use of the module has focused on discussion of the exercises. The reading material is elementary and can be understood by students without the aid of an instructor's lecture. However, as the suggested solutions indicate, the exercises require non-trivial consideration of the ideas presented and offer opportunity for thoughtful and playful speculation about how mathematics can be useful in individual problem solving.

Instructors can aid their students who use this module by providing opportunity for detailed discussion of proposed solutions. In addition, class group development of a measurement scale that aids in the solution of a problem of interest to the group can be very effective. For example, a discussion of how to measure student learning in a given course could be appropriate.

1. INTRODUCTION

Lynn just returned from her first date with Paul. Her dormitory friends gathered around. Questions came quickly from all directions. "Did you have fun?" "How did you like him?" "Will you go out with him again?"

Lynn collected her thoughts for a moment and replied, "On a scale of 1 to 10, Paul rates a 6." Some of the questioners were satisfied. Others kept on. "What did you like about him?" "What didn't you like?" "Did you like him better than Ben?"

Measurement scales, like the one Lynn invented to answer the questions fired at her, are widely used. They provide a simple way of summarizing information too complex to relate otherwise. But how useful are they in making meaningful comparisons and drawing believable conclusions?

In the material to follow, we discuss the assignment of numbers - to people, objects, or phenomena - for the purpose of measurement. Four important types of measurement scales are discussed and their limitations examined.

2. TYPES OF MEASUREMENT SCALES

2.1 Nominal Scales

In the simplest cases, assignment of numbers to people or objects serves only an identifying purpose. For example. football players wear numbered jerseys so that fans can easily recognize them on the field. Individuals have social security numbers that serve as identification for income and taxation purposes. Regions have zip codes to identify the path that mail should take in its trip from one individual to another. But these identifying numbers do not enable us to attach any meaning to the numerical order. It does not make sense to infer from the numbers alone that "Player #77 is better than Player #11," or that "The person with Social Security #284-35-2176 is older than the person with #151-32-0153," or to make other similar comparisons. This type of assignment of numbers is referred to as a nominal measurement scale. The number serves as a name for the object to which it is assigned.

2.2 Ordinal Scales

Often, however, we assign numbers to objects in such a way that we can place the objects in an ordered list. Football players may be ranked in order of their value to their team. People may be ranked according to their age or income. When numerical assignments allow us to put objects in order, based on a certain characteristic, we say that we have an ordinal measurement scale.

Suppose Lynn, whom we met above, has dated four men recently: Ben, Fred, Paul and Stan. She rates them, on her scale of 1 to 10 (where 1 is the lowest possible rank and 10 means "perfect"):

Ben	:	9
Fred	:	6
Paul	:	6
Stan	:	3.

Lynn's assignment of numbers to her dates is an example of an ordinal measurement scale. The numbers communicate her order of preference for the four men. From her list it is valid to conclude that Lynn likes Ben best of the four; she likes Paul better than Stan; she is indifferent between Fred and Paul. It is not meaningful to conclude that Lynn's liking for Ben is "three times as great" as her liking for Stan or that the amount of her preference for Ben over Fred is "equal to" the amount of her preference for Fred over Stan. The difficulty is this: we have not defined standard units -- like centimeters or liters -- for "liking." Neither subtraction nor multiplication of preferences is meaningful.

When we rank candidates in a given election, we express our preferences on an ordinal scale. For example, in the 1980 primary struggles, we might have ranked several of the Republican contenders as follows:

Anderson	:	1 (First choice)
Bush	:	2 (Second choice)
Connally	:	4 (Fourth and last choice)
Reagan	:	3 (Third choice).

Here, in contrast to Lynn's ranking, smaller numbers are associated with higher preferences. The context for a given measurement situation will indicate whether a large or a small number indicates the top of the ordering.

A different illustration of the ordinal measurement scale is the ranking of events in time. For example, if considering the American Revolution, the Civil War, the Spanish-American War, and the War of 1812, we can determine the following order:

American Revolution	:	1
Civil War	:	3
Spanish American War	:	4
War of 1812	:	2.

2.3 Interval Scales

Most students of history know more about the four wars mentioned above than simply the order in which they occurred. Uniform intervals - minutes, hours, years - have been established for measuring time. For example, counting in years from the birth of Christ, we may associate the following year-numbers with the start of each of these wars:

American Revolution	: 1775
Civil War	: 1861
Spanish American War	: 1898
War of 1812	: 1812.

Because this assignment of numbers to the wars makes use of uniform intervals - in this case, years - we can gather more meaning from the numerical values than simply their order. The differences between numerical values also have meaning; this was not the case when we ordered the wars using 1,2,3,4.

If the numbers that are assigned to objects are multiples of some fixed unit of measurement then the resulting scale is a special type of ordinal scale called an <u>interval</u> scale. On an interval scale, the <u>difference</u> between the measures - as well as their order - is meaningful.

Above, when we associated with each war the year-number in which it started, we obtained an interval scale. Because we have a standard interval of measurement, in this case a year, we can say, "The time lapse between the beginnings of the American Revolution and the War of 1812 (37 years) was the same as the time lapse between the beginnings of the Civil and Spanish-American Wars." It also makes sense to say, "A longer time passed between the War of 1812 and the Civil War than between the Civil and Spanish-American Wars." For our time measurements we have used a fixed unit of measurement, the year. For our ordinal rankings we did not have such an interval and the differences between numerical assignments could not be compared.

The thermometer that we look at daily to learn the temperature provides an additional example of an interval scale. One of the standardized intervals commonly used is a Celsius degree. Suppose that last Monday's high temperature was 16^{o} Celsius, Tuesday's high was $20^{o}C$, and Wednesday's high was $10^{o}C$. We can rank the days in order of warmth and we can also sensibly compare temperature differences. The difference between Monday's and Wednesday's warmth (6°C) is greater than the difference between Monday's and Tuesday's warmth (4°C).

A comparison that is tempting to make, for the temperatures given above, is to say that last Tuesday was twice as warm as last Wednesday. This comparison is not meaningful! The difficulty occurs because the 0° point on the Celsius scale does not indicate the absence of all warmth. The temperature is s Celsius is simply a reference point on the measurement scale. It is the temperature at which water freezes. Negative temperatures also are possible; for example, 0°C designates a warmer day than does -14°C.

Suppose John is 20" taller than his little brother Al and his sister Cindy is 10" taller than Al. We cannot conclude from this information that John is twice as tall as Cindy, since the heights are given with reference to Al's height. The case of comparing Celsius temperature is analogous to this.

2.4 Ratio Scales

When we assign numbers to objects it is desirable to be able to identify a situation that has none of a particular characteristic and to assign to it the numerical value zero. In assigning temperatures this is accomplished by use of the Kelvin scale. On the Kelvin scale, temperature measures begin at a point referred to as "absolute zero." This is the temperature at which all particles whose motion constitutes heat would be at rest. Its location is approximately 273° below zero on the Celsius scale. The temperature 0°K describes the absence of all heat. Thus it is meaningful to say that an object whose temperature is 20°K is "twice as warm" as one whose temperature is only 10°K.

When we have an interval scale with the additional property that the absence of the measured characteristic is assigned the value zero, then the scale of measurement we obtain is called a ratio scale. The Kelvin temperature scale is a ratio scale.

Use of ratio scales is not limited to the scientific laboratory, however. There are a number of common ratio scales. Heights and distances are usually measured using ratio scales. Earlier we mentioned John, Al and Cindy. Using the inch as a unit of measure, we may measure and find that John is 60" tall, Cindy is 50" tall and Al is 40" tall. Our choice of a fixed unit of measurement, the inch, gives us an interval scale. Our implicit use of zero to designate no height makes our scale a ratio scale. (Note the contrast here with the earlier case when Al's height was used as the base for comparison.) The following statements are meaningful: John is 1 1/2 times as tall as Al.

If Al grows to be twice as tall as his present height, his adult height will be 80". When ratio scales are used it **is** sensible to form ratios or multiples of measurements.

3. HISTORICAL DEVELOPMENT OF A MEASUREMENT SCALE

Temperature provides an illustration of how measurement can progress from simple to more complex and useful scale types. We can conjecture that early societies probably distinguished only between cold and warm, thus using only a nominal scale. Later, comparisons were introduced. Days were described as warmer or colder than other days. Fire was warmer than sunshine-produced temperatures. Nights were cooler than days. Temperature measurements like these used an ordinal scale. Later, during the 17th and 18th centuries, the need to record and compare the results of scientific experiments led to the invention and perfection of thermometers, which employ an interval scale of measurement with units called degrees. In the 19th century the British Physicist William Thomson (Lord Kelvin), in his work in thermodynamics, introduced the absolute scale of temperature that now bears his name. As noted above, the Kelvin Scale is a ratio scale.

4. WHY ALL THE INTEREST IN MEASUREMENT SCALES?

For most of us, measurement seems to be something we can do effectively without knowing what type of measurement scale we're using. We question the worth of discussion of scale type. "If it works, use it," is our common reply.

While not refuting the quoted reply, the following reason suggests caution in the interpretation of numerical measures:

> Many consumers of numerical information try to deduce more information from measurement scales than is actually given.

For example. suppose the City Council is considering three alternative desegregation plans. For convenience, these three proposals have been labeled Plan 1, Plan 2 and Plan 3. Plan 1 may receive greater favor because of its label - which could be (subconsciously, perhaps) treated as a ranking instead of simply a name.

A well know phenomenon in elections is the effect that the order of names on the ballot has on the outcome of an election. If a primary ballot listed four candidates for the office of mayor and all were equally known and liked, it would be expected that the person whose name appeared first would get the most votes and the person whose name appeared last would get the second highest number. Voters who are not committed to a particular candidate seem to treat ballot position as an ordinal scale that ranks the candidates.

In 1977, The Peoples' Almanac asked 35 U.S. sports writers: "In your opinion, who were the 15 greatest male athletes from 1900 to 1977?" The results, ranked in order, were:

1. Jim Thorpe
2. Babe Ruth
3. Muhammad Ali
4. Jack Dempsey
5. Jack Nicklaus
6. Ty Cobb
7. Bobby Jones
8. Joe Louis
9. Jesse Owens
10. Red Grange
11. O.J. Simpson
12. Jackie Robinson
13. Hank Aaron
14. Arnold Palmer
15. Mark Spitz

It is tempting to draw unjustified conclusions about this ordering. For example, one might comment that Mohammad Ali is "near the top." However, a further examination of the experts' ranking revealed that Jim Thorpe recieved 287 points, Muhammed Ali received 157 points and Mark Spitz 40 points. Thus, in terms of points tallied, Ali proved nearer to Spitz than to Thorpe. (To tabulate the results of this survey, vote counters assigned 10 points to each first choice, 9 to each second choice, and so on -- with athletes ranked below 10 getting no points for that ranking.)

When numbers are assigned to people, objects, or phenomena, it is tempting to apply what we know about numbers in other situations -- and to reach wrong conclusions. Often, in fact, it requires a great deal of fortitude to avoid this.

Alert consumers of numerical information do not allow themselves to read more information from a measurement scale than is actually given. They remind themselves that one can't "make a silk purse from a sow's ear" or "get blood from a turnip." Numbers are not magical devices that bring truth from the void!

5. COMPARING MEASUREMENTS

A popular way of denying the validity of a certain comparison is to say, "It's like comparing apples and oranges." Suppose we are asked the question, "Which is more -- three apples or two oranges?" A good response to this question is to ask, "On what basis are they being compared?"

We <u>can</u> compare three apples and two oranges if we compare them in terms of a particular unit of measure. We can weigh them and find out which is heavier -- using the ounce as our unit of measure. Or we can find out which costs more -- using cents as the common measure. Other measures also are possible. A medium orange (2 5/8 inches in diameter) yields 65 calories of food energy and 66 milligrams of Vitamin C, while a medium apple (2 3/4 inches in diameter) yields 80 calories of food energy and 6 milligrams of Vitamin C. Thus in caloric content three apples provide "more" than two oranges. In Vitamin C content, the reverse is true.

In summary, the apple-oranges controversy yields the following rule for comparing measurements:

> Measurements may be meaningfully compared when expressed in terms of the same units.

6. USING MEASUREMENT SCALES IN DECISION MAKING - UTILITY

The technical term used in assessing the value, satisfaction, or happiness that a particular outcome brings is sometimes called <u>utility</u>. Alternatives are compared by estimating how many "utils" of happiness are associated with them.

For example:

> Teresa is choosing between two part-time jobs. Job A pays $4 per hour but is relatively dull. Job B pays $3 per hour but is more fun. She ponders, "Which job shall I choose?"
>
> A restatement of Teresa's problem is this: to measure the dollar value and the enjoyment value in similar units so that the two jobs may be compared (otherwise it is the old problem of whether three apples or two oranges is "more"). Utils provide the common measurement unit.
>
> Let us suppose that Teresa has given the problem some thought and has decided on a unit of happiness. She can't describe it to us exactly, but we all agree to call it a "util."
>
> Teresa examines her values and decides that $1 (the money difference between the two jobs) is worth 2 utils of happiness, whereas the boredom difference

between the jobs is 3 utils. As a consequence of her analysis, Teresa chooses Job B.

Clearly- our own individual assessment of the relative value of money and fun need not agree with Teresa's. In fact, Teresa's own assessment may change over time. Faced with a college tuition increase or other inflationary pressures, Teresa may adjust her utility assignments.

Assignment of utilities to outcomes is a confusing process. Some argue, "How can I assign utility values unless I know my preference already? And if I know my preference clearly, what good are utils in making my decision?"

To answer these questions, we may say simply that the exercise of trying to assign numerical values to preferences is an effort that requires us to think clearly about those preferences and to prioritize them. Striving to measure the utility of various alternatives -- that is, defining a util and measuring our likes and dislikes in terms of it -- is disciplined thinking. Thus it is of value.

7. SUMMARY

We have discussed four types of measurement scales:

Nominal -- a measurement scale on which each object is assigned a number that serves as a name or label for the object.

Ordinal -- a measurement scale on which objects are ranked in a certain order. Each object is assigned a number that indicates its position in the ranking.

Interval -- a measurement scale for which there is a fixed unit of measurement. The numbers assigned to objects are multiples of that fixed unit. With an interval scale, not only are objects ranked, as with an ordinal scale, but also differences in rankings may be meaningfully compared.

Ratio -- a measurement scale for which there is a fixed unit of measurement and on which the number zero denotes the absence of the characteristic being measured. Objects are assigned numbers that are multiples of the fixed unit. With a ratio scale, objects can be ranked (as with an ordinal scale), differences in rankings may be meaningfullly compared (as with an interval scale) and, in addition, ratios of rankings may be meaning-fully compared.

When an individual introduces a new type of measurement, it is common to start -- as Lynn did, in evaluating her dates -- with an ordinal scale. As more information gathering and analysis are put into a particular measurement effort, it is usual to try to define a unit of measurement and thus develop an interval or ratio scale. For example, if Lynn were able to define for herself a unit of pleasure or "util," then she could rate each date (both past and future) in terms of the number of "utils" that she gained from the experience.

Efforts at defining a "util" can be especially beneficial for individuals who face a choice between alternatives whose characteristics are not all measured in terms of the same units -- as the example of Teresa's job choice indicates.

In group decision making, however, it is far more difficult to find ways of accurately incorporating individual preference intensities into the determination of a consensus. To define a "util" that has the same meaning for more than one individual is an extremely difficult, if not impossible, task. Typically, this difficulty is simply ignored. Elections are usually conducted by permitting voters to cast only one vote for a candidate regardless of the strength of their preference for the candidate. If Mark strongly supports one candidate while Nancy mildly prefers another, they both indicate their preference by casting a single vote -- and the election results produce no evidence of the amount of their support.

In a number of decision situations that involve ranking of alternatives -- for example, the sports poll discussed above -- voting participants assign points to their ranked choices. Again, such a scheme cannot be expected to provide an accurate measure of intensities of preferences. One voter may feel a strong distinction between his first and second choices, for example, while another voter may be nearly ambivalent.

Despite the weaknesses that are readily apparent in decision making methods, since decisions must continually be made, imperfect methods must be used.

8. EXERCISES

1. (a) If runners in a marathon are "measured" according to the position in which they finished (i.e., first, second, ...), what type of measurement scale is being used?

   (b) If runners in a marathon are "measured" according to the length of time it took them to complete the race, what type of measurement scale is being used?

   (c) If Ted finished 200th in the 1980 Boston Marathon and Roger finished 400th, would it be meaningful to say that Ted ran the Marathon twice as fast as Roger?

   (d) If Janet finished the Boston Marathon in 3 hours and Paula finished it in 6 hours, would it be meaningful to say that

Janet ran the Marathon twice as fast as Paula?

(e) Explain why the answers to (c) and (d) differ.

2. Karen, Leon and Murray just learned of their scores on the math test they took last Friday. Karen's score was a 95 out of 100 possible points, Leon's was 70, Murray's was 35. Which of the following are reasonable conclusions to draw from this information?

   (a) Leon knows twice as much mathematics as Murray does.

   (b) Karen got more test answers correct than Leon did.

   (c) The amount of mathematics that Leon knows is closer to the amount that Karen knows than it is to the amount Murray knows.

   (d) Think about: just what does a mathematics test (or any other test for that matter) measure? How meaningful are tests as measures of what an individual knows?

3. (a) When grades (such as A,B,C,D,F) are assigned to a student at the end of a course, what type of measurement scale is involved? Is there a difference if numerical values (such as 4,3,2,1,0) are used?

   (b) Is the grade point average a meaningful quantity? Discuss.

4. A mother, administering a punishment to her child, was heard to say, "This hurts me more than it hurts you." What assumptions is the mother making about measurement scales for pain when she says that? Is her statement a meaningful one?

5. An argument against the "one-man-one-vote" principle is that the vote allows no indication of strength of preference. One candidate can be elected over a second if 51% of the voters mildly prefer the first while 49% vigorously dislike the first and strongly prefer the second. Discuss this problem. What, if anything, can be done about it?

6. Under what conditions (if any) is it meaningful to say:

   (a) I like you twice as much as I like him?

   (b) It's twice as warm today as it was on December 31?

   (c) Albert is smarter than Thomas?

   (d) The Celtics are a better basketball team than the 76ers?

7. What units of measurement would be appropriate to use to compare:

   (a) Distances by air from Washington, DC to other cities.

   (b) Popularity of US Presidents.

   (c) Excellence of baseball teams.

   (d) Size of persons.

   (e) Intelligence of persons.

8. Average per capita income in Florida was $3700 in 1970 and $6700 in 1977. Is it valid to conclude that the average Floridian was better off in 1977? Discuss briefly.

9. Consider the following example of a "clever" host who is about to entertain two guests. He wishes to serve either coffee or tea -- whichever the pair of guests prefer -- but not both. As he plans ahead, it occurs to him that one guest might prefer coffee, the

other tea, and he would not know which to serve. He devises the following scheme. He will ask both guests to rank the following seven beverages; on the basis of their rankings he will choose between tea and coffee. The seven beverages are:

Coffee	Coke
Milk	Hot Chocolate
Seven-Up	Orange Drink
Tea	

When the guests arrive, they indulge the eccentricities of their host and provide the following preference lists (with preferred beverages higher on the list):

Guest A	Guest B
Tea	Coffee
Coffee	Coke
Hot Chocolate	Seven-Up
Milk	Orange-Drink
Orange Drink	Milk
Seven-Up	Hot Chocolate
Coke	Tea

When the host surveys the pair of preference lists, he decides to serve coffee. On what basis does he make that decision? Formulate a valid criticism of his method. Can you find a better way to solve his dilemma of whether to serve coffee or tea?

10. A school committee of five members is evaluating desegregation plans labeled A, B and C. Individuals rank the plans as follows:

Member 1	Member 2	Member 3	Member 4	Member 5
A	B	C	C	A
B	C	A	B	C
C	A	B	A	B

Based on the given information, which plan does the committee as a whole prefer? How did you decide on their preference?

11. List the courses you are taking this semester.
    (a) Interesting. Rank these courses in order from the most to the least interesting.
    (b) Worthwhile. Rank these courses in order (best to worst) based on the criterion "worthwhile for the future."
    (c) Easy. Rank these courses in order from the easiest to the most difficult.
    (d) Use the results of (a), (b) and (c) to order your courses in an overall ranking from "best" to "worst." What problems are encountered as you try to form this overall ranking?

12. As you complete the following items, note the questions that arise and the difficulties that you encounter:

(a) List at least five activities that you really like to engage in.

(b) Order the list, with the more preferred items near the top and less preferred items near the bottom.

(c) Define a unit of measure of "liking" -- you could call it a "util" -- and give the "size" of each item on your list as measured in terms of your unit of measure.

What difficulties did you encounter in completing (a), (b) and (c)? How did you overcome them?

13. Consider the following four problems:

(a) selecting a career

(b) obtaining a college degree

(c) formulating a budget

(d) managing time.

Select one of these problems that is interesting and important to you. Make a list of choices that are available within the problem. Also list the criteria you will use to select the best choice.

Compare this with Teresa's problem: Her choices were Job A and Job B, her criteria were money and fun.

If, for example, you are focusing on the problem of obtaining a college degree, your first hurdle may be to decide on a major subject. Your alternatives might be: Accounting, Biology and Mass Communications. Criteria for judging these alternatives might be: the difficulty of the course work and the probability of finding a good job after obtaining a degree in that field.

To start your analysis it would be best to consider only two or three choices and two or three criteria. Once these have been analyzed you will see more clearly how to work with expanded lists of choices and criteria.

Develop a subjective unit of measure -- call it a "util". Evaluate each of your possible choices according to the total number of utils that it provides. Rank the choices in priority order. Reflect on the results. What have you learned from your analysis? How can you improve it?

9. SOLUTIONS TO EXERCISES

1. (a) An ordinal scale.

(b) A ratio scale.

(c) The given statement is not meaningful. We could, however, say that Ted ran the Boston Marathon faster than Roger did.

(d) The use of the hour as a unit of measure gives us a ratio scale. Thus, we could say that Janet finished the Marathon in half the time that Paula did. From this, since both ran the same distance, we may conclude that Janet ran, on average, twice as fast as Paula did.