Mathematical Concepts from Unit I for Review

Some Set Theory

A *set* is a collection of some sort that has the property that, given any object, we can tell for certain whether or not that object is in the collection.

A set can have many members (*elements*), or it can have none. We write $x \in A$ if we want to say that x is an element of A. If we want to say that x is not an element of A, we write $x \notin A$. Sets that have elements we will label using upper-case letters, like A, B, C, etc. The set with no elements is called the *empty set*; it is denoted by the symbol $\emptyset$.

We can define sets by listing all of their members and surrounding them with curly braces; for instance, {lamp, chair, desk} is the set of my office furniture. We can also define sets by giving a property that all the elements of the set must satisfy. For instance, the set {x | x is an even positive integer} is equal to the set {2, 4, 6, 8, ….}. The symbol "|" in the description of the set above means "such that"; the entire statement should be read, "The set of all x such that x is an even positive integer." This way of writing things will save time and ink if the set is a large one.

Note that, in the previous paragraph, we assumed that two sets are equal if they have the same elements. This is the definition of equality for sets.

The most important feature of sets is that, given anything, we can tell for sure whether or not it is in a given set. For instance, the set of all things I have eaten today (viz., {granola, ham sandwich, yogurt} has this property, while $\{x \mid x \notin x\}$ is not. (To see why, ask yourself, "Is this set an element of itself or not?" This question can't be answered – so this is not a set!) Don't worry too much about these "pathological" entities. For everything we do in this class, we can assume that any collection of things under consideration will be a set.

If every element of a set A is a member of a set B, then we can write $A \subseteq B$ (read, "A subset B"). Note that the converse statement is not true: not every member of B is a member of A. If, on the other hand, the converse is true (i.e., $A \supseteq B$) then it is clear that A = B.

We can also define logical operations on sets: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$ and $A \cup B = \{x \mid x \in A \text{ or } x \in B, \text{ or both}\}$.

Using these definitions, we can prove some simple theorems about sets, e.g. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. It is possible to give a graphical proof using *Venn Diagrams*. It is also possible to do it analytically, by showing that the sets on either side have exactly the same elements as each other; this uses the fact that two sets are equal if they have the same elements as each other.

An analytic proof of the theorem above would read as follows: Suppose $x \in (A \cap (B \cup C))$. Then it is in A and it is also in $(B \cup C)$, so it is either in A and B, or it is in A and C, or it is in both. So it is in $(A \cap B)$ or $(A \cap C)$ or both. But this just means that it is in $(A \cap B) \cup (A \cap C)$, thus demonstrating that $(A \cap (B \cup C)) \subseteq (A \cap B) \cup (A \cap C)$. Now consider some $x \in (A \cap B) \cup (A \cap C)$. Then x is in $(A \cap B)$ or it is in $(A \cap C)$, or both, so there are three possible cases. Suppose it is in $(A \cap B)$. Then it is in both A and B, so it must be in $A \cap (B \cup C)$. A similar argument holds for the other two cases, so we have shown that $(A \cap B) \cup (A \cap C) \subseteq (A \cap (B \cup C))$. Therefore, the two sets are equal.

Relations and Functions

The relations we are most familiar with are *binary relations*, i.e., relations between two objects. We say things like, "I prefer chicken to asparagus" and "The length of this plutonium slug is 1.02997 cm.". In the first case, we are expressing a relation between pairs of items on the menu of a restaurant; in the second, we are expressing a relation between a set of objects (plutonium slugs) and the positive real numbers. Two things are being associated in each instance. It is possible to consider relations between three or more objects, but we won't need these for studying decision theory.

The first type of relation we considered above – preference – is an example of a *relation in a set*. Members of a set are being compared with each other. This stands in contrast to the second relation, which is a *relation between sets*. It compares items in one set (viz., the set of plutonium slugs under consideration) with items in the other (the positive real numbers).

The notion of an *ordered pair* is essential to an understanding of relations. We write ordered pairs as (x, y) (not the curly braces used for sets). While for sets, {a, b} = {b, a} for any b and a, for ordered pairs, the statement (a, b) = (b, a) can only be true if a = b. We can extend the notion of ordered pairs to ordered triples (e.g., (x, y, z)) or, more generally, to ordered n-tuples, $(x_1, x_2, \ldots. x_n)$.

The set of all ordered pairs were (a, b) where $a \in A$ and $b \in B$ is called the *Cartesian product* of A and B, and is denoted $A \times B$. The product $A \times A$ is sometimes written as A^2.

A *relation in (or on) a set* A can be formally defined as a subset of A^2. A *relation between two sets* A and B can be formally defined as a subset of $A \times B$. If x and y are two elements, we say that "x is related to y" or "xRy" if the ordered pair (x, y) is an element of the relation.

We can represent relations by tables. For instance, to represent a relation between A and B (that is, the relation that is a subset of $A \times B$), list the elements of A in rows in the margin of our paper, and the elements of B as columns running across. The relation can be specified by marking the boxes created by the rows and columns. For instance, to express the relation xRy, we would mark the box on the same row as x and in the same column as y.

Many names have been given to the properties of relations in sets. In class, we discussed *symmetry* (xRy implies yRx), *transitivity* (xRy and yRz implies xRz), *reflexivity* (xRx for all x) *completeness* (in the relation, xRy or yRx or both), and *equivalence* (reflexivity, symmetry, and transitivity taken together).

When describing relationships between sets, it is often useful to use a special kind of relation called a *function*. A function is a relation such that, if (x, a) and (x, b) are both in the function, then a = b. What this means is that, to each value of the first entry in the ordered pair, a *unique* value is given (the second entry in the ordered pair) – functions don't have "multiple outputs". In the relation (mentioned above) between plutonium slugs and the real numbers, the condition for a function is satisfied – after all, a single slug cannot have multiple lengths! Note on the other hand that it is permitted for a particular length to have several slugs associated with it; after all, there could be different slugs with the same length. Functions are thus very common in situations where we are assigning measurements to a set of items.