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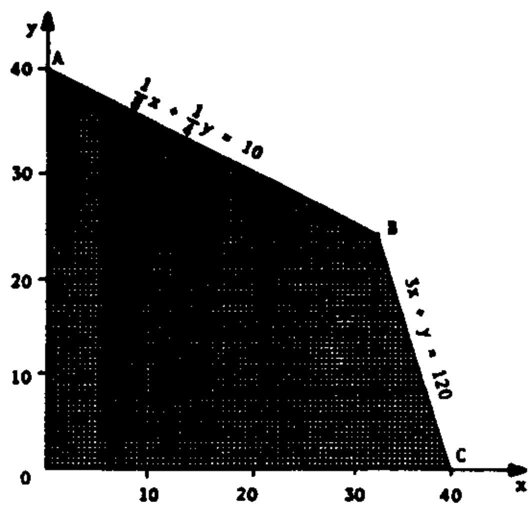
Module 453

Linear Programming in Two Dimensions: I

Nancy S. Rosenberg



Applications of High School Algebra to Operations Research

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Modules and Monographs in Undergraduate Mathematics and Its Applications (UMAP) Project

The goal of UMAP was to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications to be used to supplement existing courses and from which complete courses may eventually be built.

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LINEAR PROGRAMMING IN TWO DIMENSIONS: I

by

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Intermodular Description Sheet: UMAP Unit 453

Title: LINEAR PROGRAMMING IN TWO DIMENSIONS: I

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Math Field: High school algebra

Application Field: Operations research

Target Audience: Students in finite mathematics and quantitative business administrative courses, high school juniors and seniors.

Abstract: This module introduces a variety of linear programming problems and describes in detail a graphical method for solving linear programming problems in two dimensions. Students formulate simple linear programming problems and solve them graphically.

Prerequisites: Be able to graph and solve simultaneous sets of linear equations in two unknowns.

1. LINEAR PROGRAMMING PROBLEMS

1.1 Examples of Linear Programming Problems

Several years ago, a major grain supplier decided to produce chicken feed from a mixture of grains and food supplements. Each of the possible ingredients had a different price, and each contained different proportions of the various nutrients that chickens need each day. The question was this: Which ingredients, in which proportions, should be combined to meet the nutritional needs of the chickens as inexpensively as possible?

The producers of a Broadway musical were designing an advertising campaign. They planned to advertise through several different media. Each type of advertisement was known to reach different numbers of people in various income brackets, and each had a different cost. The producers knew how many people they had to reach in each income bracket if the campaign were to be successful. How should they distribute their advertising dollars among the various media in order to have an effective campaign at the minimum possible cost?

A farmer planned to grow several crops, each of which required different amounts of irrigation and acreage. In addition, the labor costs associated with each crop were different, as were the selling prices. Naturally, the farmer had limited amounts of water, land, and capital available. How much of each crop should she plant in order to maximize her profits?

1.2 The Characteristics of Linear Programming Problems

What do these three problems have in common? First, they all involve quantities that can be assigned a whole range of possible values at the will of the problem solver. The grain supplier can decide which ingredients he will use and in which proportions he will use them. The

Broadway producers can choose to run different numbers of advertisements on radio and television or in newspapers and magazines. The farmer can plant varying amounts of many possible crops. These are called *controllable variables*. Second, all three problems involve conditions that limit the range of values that these variables can assume. The grain supplier must meet the nutritional needs of the chickens, the producers must reach certain numbers of people, and the farmer must stay within the limits of the available water, capital, and land. These are the *constraints*. Third, each problem has as its object the minimization or maximization of a critical quantity. The grain supplier and the producers wish to minimize their costs; the farmer wants to maximize her profits. Taken together, these are some of the major characteristics of linear programming problems.

1.3 What is Linear Programming?

Linear programming is a mathematical technique for achieving the best possible results in a situation that is governed by restrictions. It is not to be confused with computer programming, which is programming of an entirely different sort. Of the many quantitative procedures that are now used as aids in decision making, linear programming is one of the most successful. It is applicable to a wide variety of situations, and it has already helped to save many millions of dollars.

The word "linear" refers to the fact that the mathematical equations used in a linear program are equations of the first degree. In two dimensions, these are the equations of straight lines. Anyone who can graph linear equations and inequalities in a two-dimensional coordinate system and solve them simultaneously can learn to solve simple linear programming problems.

2. A SIMPLE PROBLEM IN LINEAR PROGRAMMING

2.1 Formulating the Problem

Let us return to the problem of producing an economical feed for chickens. For the sake of simplicity, we will consider just two of the feed's ingredients, corn and alfalfa. (Although the reasoning used here is similar to that used in solving real life problems, the numbers have been altered to simplify the computations.)

Suppose that corn is priced at 6¢ a pound, alfalfa at 8¢ a pound. Each pound of corn contains 2 mg of protein, 1 mg of thiamine, and 14 mg of fat. (Mg stands for milligram, a very small unit of weight. There are 1000 milligrams in a gram and 28.4 grams in an ounce.) Each pound of alfalfa contains 1 mg of protein, 5 mg of thiamine, and 25 mg of fat. Animal nutritionists have determined that chickens require, at a minimum, 15 mg of protein per week and 30 mg of thiamine. It is also known that chickens will not eat more than 285 mg of fat per week. This information is summarized in Table I below.

TABLE I

	protein	thiamine	fat	cost
corn	2 mg/lb	1 mg/lb	14 mg/lb	6¢/lb
alfalfa	1 mg/lb	5 mg/lb	25 mg/lb	8¢/lb
minimum required	15 mg	30 mg		
maximum allowed			285 mg	

Given these conditions, how many pounds of corn and how many pounds of alfalfa must be mixed together to meet the chicken's weekly requirements at the lowest possible cost?

The first step in formulating a linear programming problem is to assign symbols to the controllable variables, in this case the number of pounds of corn and the number of pounds of alfalfa that are to be used in the chicken's weekly feed.

Let x = the number of pounds of corn to be used.
Let y = the number of pounds of alfalfa to be used.

Now the constraints can be stated in terms of x and y. We will start with the protein constraint. Since each pound of corn contains 2 milligrams of protein, the number of milligrams of protein in x pounds of corn will be $2x$. In the same way, the number of milligrams of protein in y pounds of alfalfa will be $1y$, or simply y. Then the total amount of protein in the corn and alfalfa mix will be $2x + y$. And since each chicken needs at least 15 milligrams of protein every week, we know that $2x + y$ must be at least 15. In algebraic terms,

$$2x + y \geq 15.$$

Similarly, since each gram of corn contains 1 milligram of thiamine and each gram of alfalfa contains 5 milligrams of thiamine, in order to have at least 30 milligrams of thiamine in the chicken's weekly feed we must be sure that

$$x + 5y \geq 30.$$

Unlike the constraints on the protein and thiamine, which set minimum values, the constraint on the fat sets a maximum value. The fat content in the chicken's weekly feed cannot exceed 285 milligrams. Since the corn will contain $14x$ milligrams of fat and the alfalfa $25y$ milligrams of fat, it is necessary that

$$14x + 25y \leq 285.$$

It is also important to realize that neither x nor y can be negative, that is,

$$x \geq 0 \text{ and } y \geq 0.$$

Having formulated the constraints, we must state the object of the program, which is to minimize the cost of the feed. This cost will be the sum of the cost of the corn and the cost of the alfalfa. We know that x pounds of corn at 6¢/lb will cost 6x cents; y pounds of alfalfa at 8¢/lb will cost 8y cents. The total cost of the mix, in cents, will therefore be

$$C = 6x + 8y$$

where C stands for cost. Because it is our object to minimize the value of C, this equation is called the *objective function*.

The linear program for this problem is summarized below.

Letting x = the number of pounds of corn to be used
and y = the number of pounds of alfalfa to be used

Minimize $C = 6x + 8y$

subject to

$$2x + y \geq 15 \quad \text{(protein)}$$

$$x + 5y \geq 30 \quad \text{(thiamine)}$$

$$14x + 25y \leq 285 \quad \text{(fat)}$$

where $x \geq 0$ and $y \geq 0$.

Example 1. Formulate the constraints and the objective function for the following problem. A bakery must plan a day's supply of eclairs and napoleons. Each eclair requires 3 ounces of custard and 7½ minutes of labor. Each napoleon requires 1 ounce of custard and 15 minutes of labor. The bakery makes 40¢ on each eclair that it sells and 30¢ on each napoleon. If 120 ounces of custard are available, and 10 hours of labor, how many eclairs and how many napoleons should the bakery make to maximize its profits?

Step 1: Assign symbols to the controllable variables.
Let x = the number of eclairs the bakery should make.
Let y = the number of napoleons the bakery should make.

Step 2: Formulate the constraints in terms of x and y.
Since each eclair requires 3 ounces of custard, x eclairs require 3x ounces of custard. Similarly, y napoleons

require y ounces of custard. 120 ounces of custard are available, so

$3x + y \leq 120.$

Eclairs require 1/8 of an hour of labor, napoleons 1/4 of an hour. With 10 hours of labor available, this means that

$1/8x + 1/4y \leq 10.$

In addition, $x \geq 0$ and $y \geq 0$.

Step 3: Formulate the objective function.
The profit on x eclairs is .40x; the profit on y napoleons is .30y. The total profit on x eclairs and y napoleons is therefore

$P = .40x + .30y.$

Formulate linear programs for the following problems.

Exercise 1. A dry cleaning company is buying up to 30 new pressing machines and is considering both a deluxe and a standard model. The deluxe model occupies 2 square yards of floor space and presses 3 pieces per minute. The standard model occupies 1 square yard of floor space but presses only 2 pieces per minute. If 44 square yards of floor space are available, how many machines of each type should the company buy to maximize its output?

Exercise 2. The producers of a Broadway musical plan to advertise on New York City buses and on a local radio station. Each bus advertisement costs $1000; each radio commercial costs $3000. The producers want to have at least one third as many bus advertisements as radio commercials. Bus advertisements are known to reach 400 upper income families, 400 middle income families, and 500 lower income families each week. The radio commercials reach 100 upper income families, 1100 middle income families, and 100 lower income families each week. If the producers want to reach at least 2100 upper income families and 9100 middle income families and no more than 5000 lower income families every week, how should they distribute their advertising between the two media in order to minimize the cost of the campaign?

Exercise 3. A farmer has 30 acres on which to grow tomatoes and corn. 100 bushels of tomatoes require 1000 gallons of water and 5 acres of land; 100 bushels of corn require 6000 gallons of water and 2½ acres of land. Labor costs are $1 per bushel for both corn and tomatoes. The farmer has available 30,000 gallons of water and $750 in capital. He knows that he cannot sell more than 500 bushels of tomatoes or 475 bushels of corn. If he makes a profit of $2 on each bushel of tomatoes and $3 on each bushel of corn, how many bushels of each should he raise in order to maximize his profits?

2.2 Graphing the Problem

It is not hard to find pairs of values for x and y that will satisy all the constraints listed in the program formulated in Section 2.1. $x = 5$ and $y = 7$ is one such pair; $x = 8$ and $y = 6$ is another. (Try them.) The possibilities are, in fact, unlimited. The question is, which of these pairs will give the lowest possible value for C? When the problem has only two unknowns, as this one does, one way to answer this question is to make a graph.

Since $x \geq 0$ and $y \geq 0$, we shall be interested in points in the first quadrant only. This is always the case when the variables in a linear programming problem represent physical quantities that cannot be negative.

We will start by graphing the protein constraint. The line $2x + y = 15$ is shown in Figure 1, as well as the shaded region where $2x + y > 15$. The points in this region, together with those on the line, are the only ones for which it is true that $2x + y \geq 15$. These points are hence the only ones that satisfy the protein constraint.

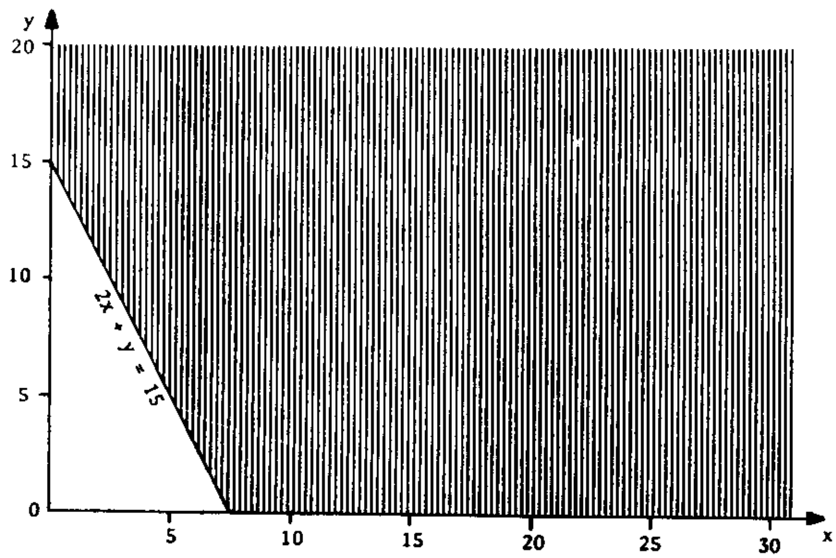


Figure 1. The points that satisfy the protien constraint.

Which of these points also satisfy the thiamine constraint? To find out, we must draw the line $x + 5y = 30$ on the same set of axes and shade in the region above it. Only those points which lie in the intersection of the two sets of points satisfy both the protein and the thiamine constraints. (See Figure 2.)

Because the fat constraint sets a maximum condition, it will be satisfied only by points on or below the line $14x + 25y = 285$. In Figure 3, this constraint is combined with the other two, and the shaded region now shows those points that satisfy all three of the constraints together. A region like this one is called a convex set. The points labelled P, Q and R are its vertices.

A set of points is convex if the line joining any two points of the set lies within the set. Convex sets have no holes in them, and their boundaries are straight or bend outward. The intersection of any two convex sets is itself a convex set.

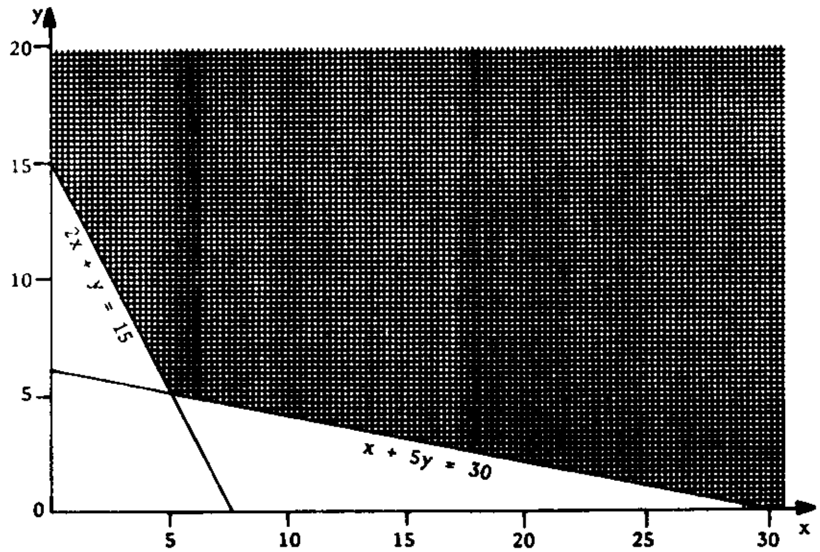


Figure 2. **The points that satisfy the protein and the thiamine constraints.**

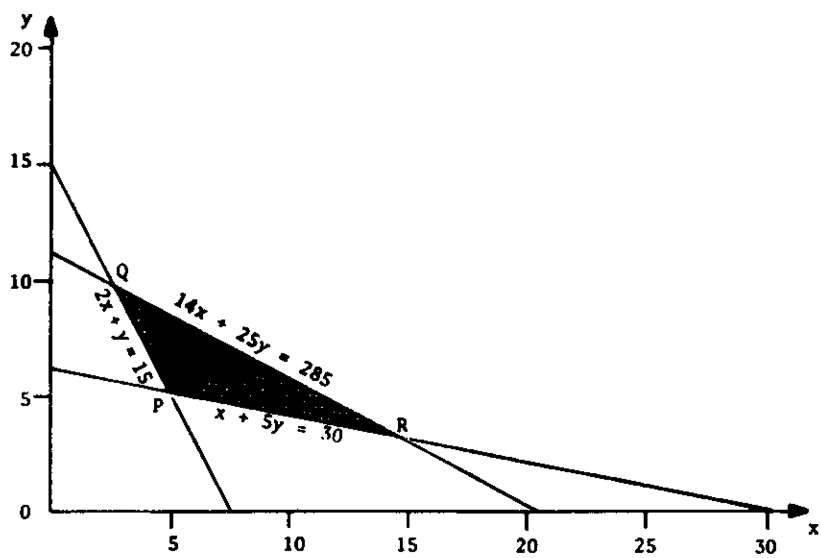
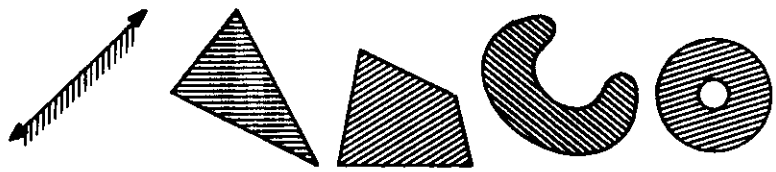


Figure 3. **The points that satisfy the protein, thiamine, and fat constraints.**

In Figure 4 below, a, b and c are convex sets; d and e are not.



(a) (b) (c) (d) (e)

Figure 4.

Set a, consisting of a straight line and all the points on one side of it, is called a half-plane. Since the triangle in Figure 3 is formed by the intersection of three such half-planes, it too is a convex set. The non-negative solutions to any linear programming problem, no matter how complex, lie in a convex set.

2.3 Solving the Problem

Now that we have a picture of all the points whose coordinates are possible solutions, we are ready to solve the problem, that is, to find the point whose coordinates minimize the cost, C, of the feed. To do this, we must interpret the equation of the objective function, $C = 6x + 8y$, as the equation of a line in the xy-plane. In slope-intercept form, this equation becomes

$$y = -\frac{6x}{8} + \frac{C}{8}.$$

Thus, the slope of the objective function is -6/8, or -3/4, and the value of C determines the y-intercept, C/8. In particular, the smaller the value of C, the smaller the y-intercept will be.

All lines with slopes of -3/4 belong to a family of parallel lines, some of which are shown in Figure 5. These lines can be viewed as possible positions of a single line with a slope of -3/4 moving across the coordinate system parallel to itself. In some of these positions it will pass through the region that is shaded

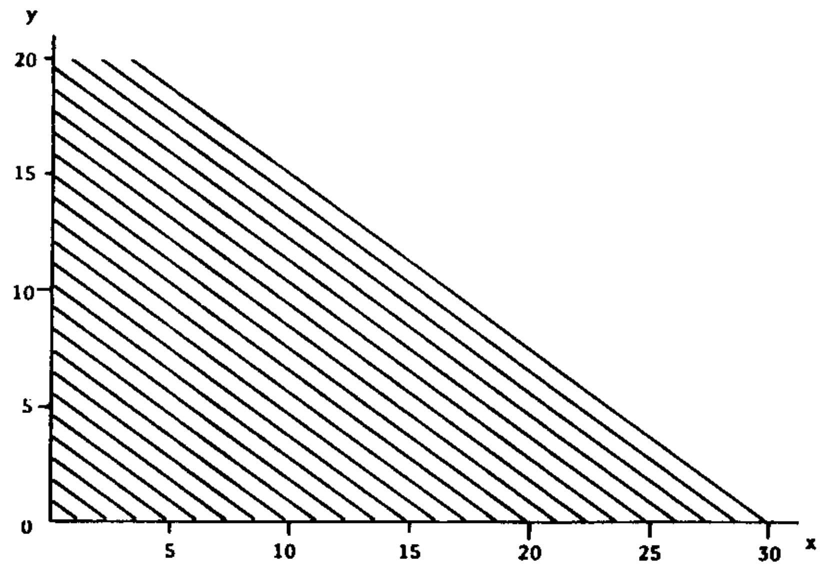


Figure 5. The family of lines whose slopes are -3/4.

in Figure 3; in others it will not. Figure 6 shows Figure 5 superimposed on Figure 3. In Figure 6, the lowest line in the family to pass through the shaded region appears to be the one that passes through the

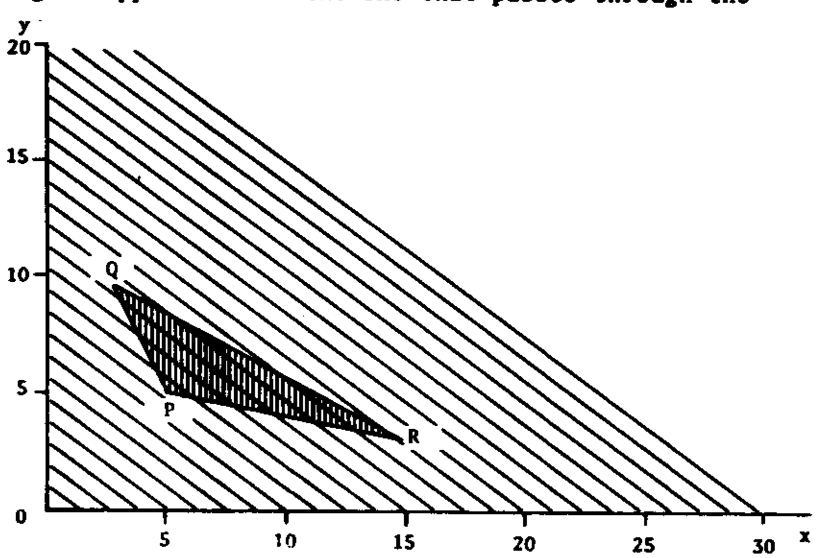


Figure 6. Figure 5 superimposed on the shaded region of Figure 3.

point marked P. Of all the lines that have slopes of -3/4 and contain at least one point that satisfies the constraints of the problem, this is the one with the smallest y-intercept. Point P is therefore the point in the shaded region whose coordinates minimize C.

It can be shown that whenever a linear equation such as the objective function $C = 6x + 8y$ is defined on a region bounded by a convex polygon, it will assume its minimum and maximum values at vertices of the polygon. To minimize or maximize the objective function of a linear program, it is therefore necessary to evaluate it only at the vertices of the convex polygon determined by the constraints. The vertex that gives the best value of the objective function is then the solution of the program. (In the special case where a side of the polygon has the same slope as the objective function, two consecutive vertices and all the points between them may minimize or maximize the function.)

Since the vertices of the convex polygon are points at which its sides intersect, the coordinates of these vertices can be found by solving the appropriate equations simultaneously. Point P is the intersection of the protein and thiamine constraints; its coordinates are therefore found by solving simultaneously the equations

$$2x + y = 15$$

and

$$x + 5y = 30.$$

In the same way, point Q is the simultaneous solution of

$$2x + y = 15$$

and

$$14x + 25y = 285$$

and point R the simultaneous solution of

$$x + 5y = 30$$

and

$$14x + 25y = 285.$$

The coordinates of P, Q and R, together with the values of C they determine, are given in Table II below.

TABLE II

	Coordinates of Vertex	Value of Objective Function at Vertex
P	(5, 5)	$C = 6(5) + 8(5) = 70$
Q	(5/2, 10)	$C = 6(5/2) + 8(10) = 95$
R	(15, 3)	$C = 6(15) + 8(3) = 114$

Table II shows that P is indeed the vertex whose coordinates minimize the objective function. Thus, the chicken's weekly feed should contain 5 pounds of corn and 5 pounds of alfalfa, and the cost of this mix will be 70¢.

To solve a linear program in two dimensions, it is therefore necessary to:

1. Formulate the constraints and the objective function.
2. Graph the constraints.
3. Shade in the convex polygon they determine.
4. Find the coordinates of the vertices of this polygon.
5. Evaluate the objective function at each of these vertices.

The vertex whose coordinates give the best value of the objective function (a maximum or a minimum as the case may be) is the solution to the linear program.

Exercise 4. Solve the linear program for Example 1, Section 2.1.

Exercise 5. Solve the linear program for Exercise 1.

Exercise 6. Solve the linear program for Exercise 2.

Exercise 7. Solve the linear program for Exercise 3.

3. CONCLUSION

The problem with which this module began was a real problem, but the version of it given in Section 2.1 was greatly simplified. When the grain supplier actually made its chicken feed, it used nearly thirty different ingredients which, taken together, fulfilled the chicken's requirements for several dozen vitamins, minerals, and other nutrients. Correspondingly, the linear program which was formulated to solve the problem contained several dozen constraints, each involving up to thirty different variables.

Like this one, most real world problems in linear programming involve a large number of variables which are subject to many different constraints. Although similar in form to those for two variable problems, their linear programs are far more complex and are usually solved by computers. The method used for their solution, however, is entirely analagous to the one presented here.

4. SAMPLE EXAM

Formulate and solve the following problems.

1. The manager of a watch company is planning a month's production schedule. The company manufactures both quartz and regular watches and wishes to produce at least as many quartz watches as regular ones. An order for 225 regular watches has already been received, but no more than 500 regular watches are sold in any one month. Quartz watches require 3 hours of production time, regular watches 2. 3150 production hours are available, and there are 1150 sets of straps on hand and 870 quartz assemblies. If the company makes $15 on each quartz watch and $7 on each regular one, how many of each should it manufacture to maximize its profits?

2. Dog food is made from a mixture of horsemeat and beef. The manufacturers want to use at least half as much horsemeat as beef and must use 75 pounds of beef already on hand. Each pound of beef contains 1 gram of calcium, 5 grams of ash, and 1 gram of moisture. Each pound of horsemeat contains 1 gram of calcium, 1 gram of ash, and 7 grams of moisture. The mixture must contain at least 225 grams of calcium and may contain no more than 1100 grams of ash and 1580 grams of moisture. If beef costs $2 per pound and horsemeat costs $1 per pound, how many pounds of each should the manufacturers use to minimize their cost?