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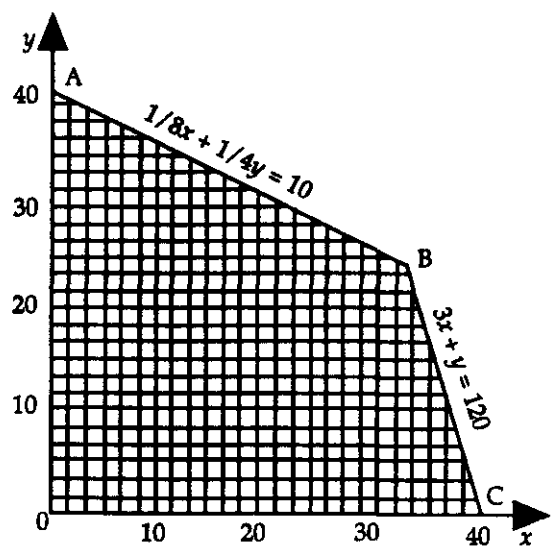
Module 454

Linear Programming in Two Dimensions: II

Nancy S. Rosenberg



Applications of Linear Programming to Operations Research

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The goal of UMAP was to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications to be used to supplement existing courses and from which complete courses may eventually be built.

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LINEAR PROGRAMMING IN TWO DIMENSIONS: II

by

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Intermodular Description Sheet: UMAP Unit 454

Title: LINEAR PROGRAMMING IN TWO DIMENSIONS: II

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Classification: APPL LIN PROG/OPER RESEARCH

Prerequisite Skills:
1. Understand what is meant by "first degree equations."
2. Be able to graph linear equations and inequalities.
3. Be able to graph and solve simultaneous sets of linear equations in two unknowns.
4. Be familiar with the contents of "Linear Programming in Two Dimensions: I" (UMAP Unit 453).

Output Skills:
1. Understand the nature of sensitivity analysis by investigating the ways in which the graphical solution of a linear programming problem is altered by the changes in various parameters.

Related Units:
Linear Programming in Two Dimensions: I *(Unit 453)*

1. INTRODUCTION

In Linear Programming in Two Dimensions: I, linear programming was presented as a mathematical technique for achieving the best possible results in situations that are governed by restrictions. In that first module, these restrictions were assumed to be certain and constant. In practice, however, they are often subject to change. New supplies of resources become available. Costs go up. Requirements are modified and estimates revised. One of the most important functions of a linear program is to determine the extent to which changes such as these will affect the solution, that is, to determine how sensitive the solution is to changes in various parameters of the problem. The study of these changes and their effects is called *sensitivity analysis*.

2. SENSITIVITY ANALYSIS

2.1 Binding and Nonbinding Constraints

Let us return to the farmer of Exercise 3 in the first module. For convenience, this exercise and its solution are repeated here.

> **A farmer has 30 acres on which to grow tomatoes and corn. 100 bushels of tomatoes require 1000 gallons of water and 5 acres of land; 100 bushels of corn require 6000 gallons of water and 2½ acres of land. Labor costs are \$1 per bushel for both corn and tomatoes. The farmer has available 30,000 gallons of water and \$750 in capital. He knows that he cannot sell more than 500 bushels of tomatoes or 475 bushels of corn. If he makes a profit of \$2 on each bushel of tomatoes and \$3 on each bushel of corn, how many bushels of each should he raise in order to maximize his profits?**

Solution:

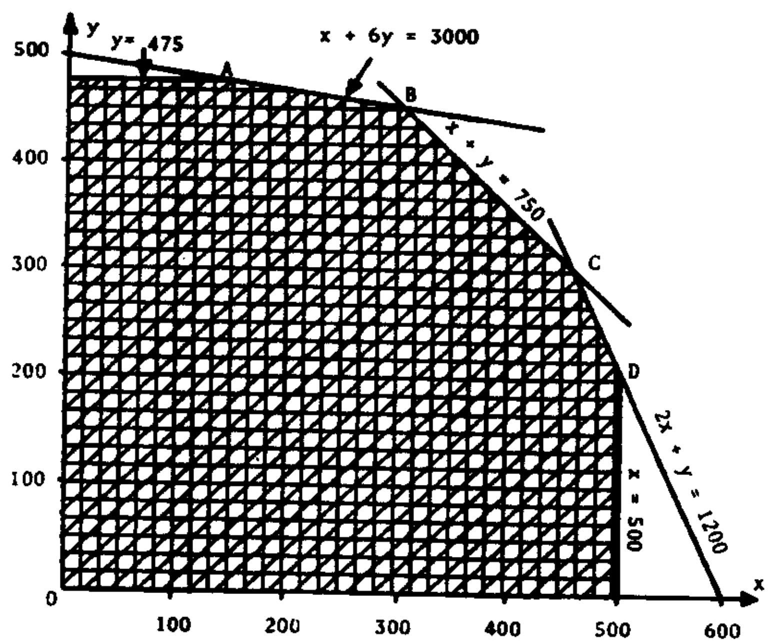
Letting x = the number of bushels of tomatoes to be raised
and y = the number of bushels of corn to be raised

Maximize $P = 2x + 3y$

subject to

$10x + 60y \leq 30000$		(water)
$(5/100)x + (5/200)y \leq$	30	(acreage)
$x + y \leq$	750	(capital)
$x \leq$	500	(limit on tomatoes)
$y \leq$	475	(limit on corn)

where $x \geq 0$ and $y \geq 0$.



	Coordinates of Vertex	Value of Objective Function at Vertex
A	(150, 475)	$P = (2)(150) + (3)(475) = 1725$
B	(300, 450)	$P = (2)(300) + (3)(450) = 1950$
C	(450, 300)	$P = (2)(450) + (3)(300) = 1800.$
D	(500, 200)	$P = (2)(500) + (3)(200) = 1600$

B is the vertex whose coordinates maximize the objective function. The farmer should raise 300 bushels of tomatoes and 450 bushels of corn. His profit will then be $1950.

The solution point, B, lies at the interesection of the water and capital constraints. Because the equality condition holds on these constraint lines, the coordinates of B must satisfy the equations $x + 6y = 3000$ and $x + y = 750$. Thus, $300 + 6(450) = 3000$ and $300 + 450 = 750$. These equations show that in producing 300 bushels of tomatoes and 450 bushels of corn the farmer will use up all the available water and capital. If he had more water or more capital, he would be able to increase his production. Because they limit the amounts of corn and tomatoes that the farmer is able to produce, these constraints are said to be *binding*. The acreage constraint, on the other hand, is not binding. Of the 30 available acres, only $(5/100)(300) + (5/200)(450)$ or 26¼ will be used. Graphically, this means that point B lies under the acreage constraint line, in the region where the inequality sign applies.

2.2 Shadow Prices

The amounts of corn and tomatoes that the farmer is able to produce are limited by the availability of water and capital. Suppose that 1000 additional gallons of water were available. The water constraint would then become $10x + 60y \leq 31000$, or $x + 6y \leq 3100$. The original water constraint, $x + 6y \leq 3000$, had a slope of $-1/6$ and a y-intercept of 500. This new water constraint also has a slope of $-1/6$, but its y-intercept is 516 2/3. Thus, the water constraint has moved parallel to itself in an upward direciton. The shaded area is larger than it was before, point B is higher, and the value of the objective function at B is correspondingly greater.

In Figure 1, the area enclosed in the square has been enlarged. B_1 shows the original position of B, B_2

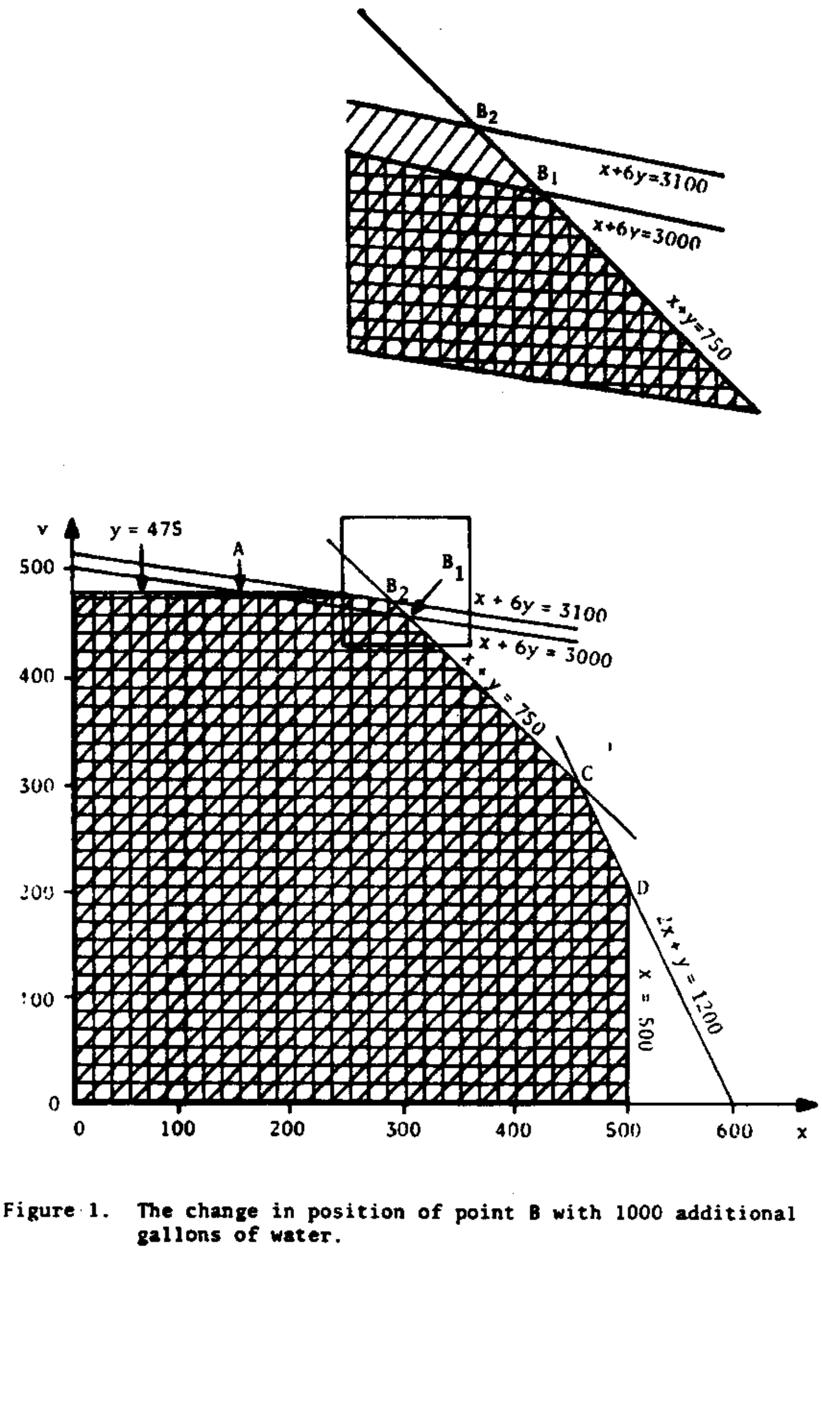


Figure 1. The change in position of point B with 1000 additional gallons of water.

the position B assumes when 1000 additional gallons of water are available. The new coordinates of B, found by solving simultaneously the equations $x + y = 750$ and $x + 6y = 3100$, are $x = 280$ and $y = 470$; the value of the objective funciton is now $P = 2(280) + 3(470)$ or 1970. The additional 1000 gallons of water have thus enabled the farmer to increase his profit from $1950 to $1970. Because each gallon contributes $20/1000 or $.02 in increased profits, the farmer should be willing to pay up to $.02 a gallon for additional water. This is called the *shadow price* of the water. The shadow price of a resource is defined as the change in the optimal value of the objective function that is produced by one additional unit of that resource.

Example 1. A bakery must plan a day's supply of eclairs and napoleons. Each eclair requires 3 ounces of custard and 7½ minutes of labor. Each napoleon requires 1 ounce of custard and 15 minutes of labor. The bakery makes 40¢ on each eclair that it sells and 30¢ on each napoleon. 120 ounces of custard and 10 hours of labor are now available. Should the bakery agree to buy 50 additional ounces of custard at 5¢ an ounce?

The solution to this problem, given in the first linear programming module, is reproduced below.

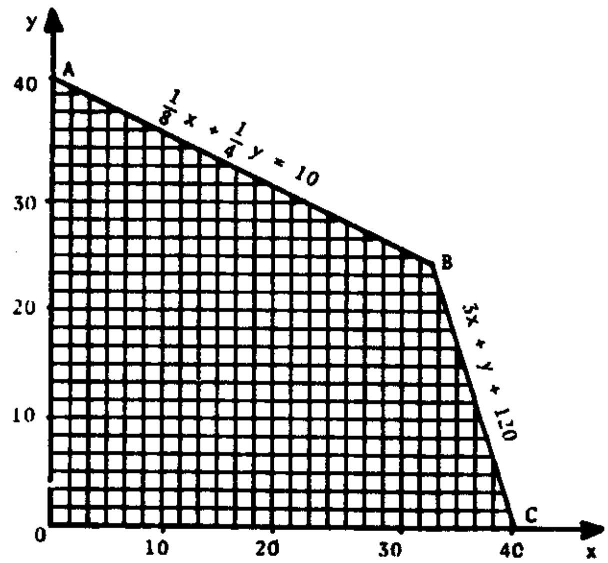
Letting x = the number of eclairs the bakery should make
and y = the number of napoleons the bakery should make

Maximize $P = .40x + .30y$

subject to

$$3x + y \le 120 \text{ (custard)}$$
$$(1/8)x + (1/4)y \le 10 \text{ labor}$$

where $x \ge 0$ and $y \ge 0$.



Coordinates of Vertex	Value of Objective Function at Vertex
A (0, 40)	P = (.40)(0) + (.30)(40) = 12.00
B (32, 24)	P = (.40)(32) + (.30)(24) = 20.00
C (40, 0)	P = (.40)(40) + (.30)(0) = 16.00

B is the vertex whose coordinates maximize the objective function. The bakery should make 32 eclairs and 24 napoleons. Its profit will then be $20.00.

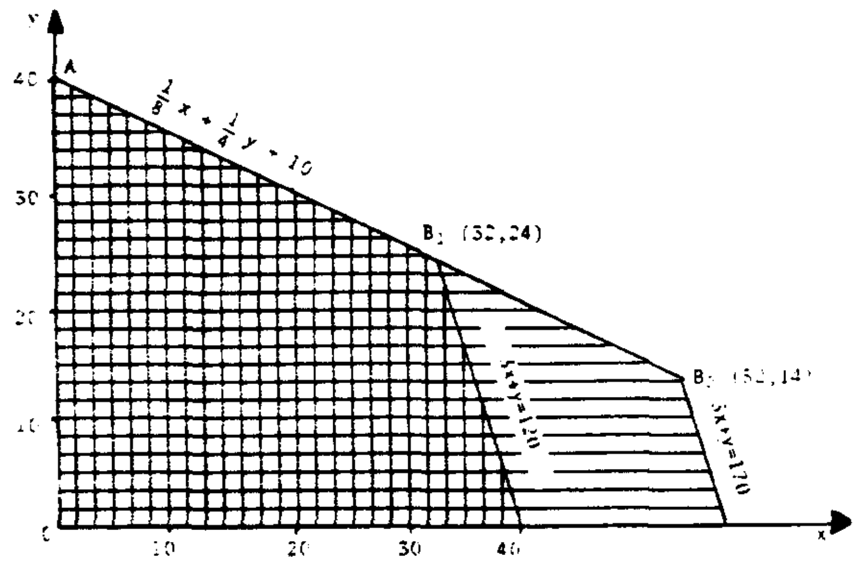
Using this solution, the shadow price of the custard will now be found.

<u>Step 1</u>: Determine whether or not the custard constraint is binding. Since B, the vertex whose coordinates maximize the objective function, lies on the custard constraint line, the custard constraint is binding. An increase in the amount of custard available will therefore produce an increase in the optimal value of the objective function.

<u>Step 2</u>: Rewrite the custard constraint. With 50 additional ounces of custard available, this constraint changes from $3x + y \leq 120$ to $3x + y \leq 170$.

Step 3: Find the new optimal solution.

As seen in the diagram below, point B, the simultaneous solution of $(1/8)x + (1/4)y = 10$ and $3x + y = 170$, now has coordinates $x = 52$, $y = 14$.



Step 4: Find the new optimal value of the objective function and use it to determine the shadow price of one ounce of custard. The new value of the objective function is $P = (.40)(52) + (.30)(14) = 25$. The 50 additional ounces of custard have therefore resulted in an increase in profit of \$5. The shadow price of the custard is thus \$5/50 or \$.10 per ounce. Since the price of the available custard is only \$.05 per ounce, the bakery should buy it.

Note: The exercises in this module are continuations of the exercises in Linear Programming in Two Dimensions: I.

Exercise 1. The dry cleaning company (Exercises 1 and 5, Linear Programming I) now has an opportunity to rent five additional machines. To pay for themselves, these machines must increase the total output by at least three pieces per minute. Should the company agree to rent them?

Exercise 2. The Broadway producers (Exercises 2 and 6, Linear Programming I) estimate that they will lose $500 for every 100 upper-income families that they fail to reach. Would it pay them to plan a campaign that reaches fewer than the 2100 upper-income families that they planned to reach originally? Should they also consider a campaign that reaches fewer lower-income families?

Exercise 3. The farmer (Exercises 3 and 7, Linear Programming I and pages 1, 2 and 3 of this module) finds that he can get a short-term loan at 15% interest. What is the shadow price of capital? Will it pay the farmer to borrow?

2.3 Limits on Shadow Prices

Binding constraints are constraints that limit the optimal value of the objective function in a linear programming problem. To improve this optimal value, it is necessary to alter the right-hand side of a binding constraint. The farmer in Sections 2.1 and 2.2 was able to make a higher profit by obtaining additional water and thus increasing the right-hand side of the water constraint. Each binding constraint has a shadow price associated with it. The shadow price is a measure of the amount of improvement in the optimal value of the objective function that is produced by one unit of change in the right-hand side of the constraint. In the case of the farmer, each additional gallon of water increased his profit by $.02.

How long can this improvement in the optimal value of the objective function be expected to continue? Would 5000 additional gallons of water cause the farmer's profits to increase by 5000 × $.02, or $100? Or would other constraints limit the extent to which additional water could be utilized?

The farmer cannot sell more than 475 bushels of corn. Each of these bushels costs him $1 in labor, and he has only $750 available. If he grows 475 bushels of corn and 275 bushels of tomatoes, the farmer will need (475)(70) + (275)(10) or 31,250 gallons of water. This is the **maximum** amount of water that he can use; beyond this he will have an excess. The shadow price of the water, therefore, applies only to 1250 gallons over and above the 30,000 he now has. This range of 30,000 to 31,250 gallons is called the *right-hand side range* for the water constraint. Outside this range, the shadow price does not apply.

2.4 Determining Right-Hand Side Ranges

When a linear programming problem is in two dimensions, the right-hand side ranges can be determined from the graphical solution. In the diagram on page 2, the water constraint is seen to intersect the corn constraint at point A and the capital constraint at point B. In Figure 2, the corn and capital constraints intersect at point P, outside the shaded area. The coordinates of P, $x = 275$ and $y = 475$, are the simultaneous solution of the equations of these constraints, $y = 475$ and $x + y = 750$.

As the amount of water available increases beyond 30,000 gallons, the water constraint moves parallel to itself in an upward direction. Eventually it will pass through point P, and when it does the amount of available water, found by substituting the coordinates of P in the left-hand side of the water constraint, is 10(275) + 60(475) or 31,250 gallons. When the number of gallons of available water exceeds this amount, the water constraint no longer passes through the shaded region, P becomes a vertex, and the corn and capital constraints are binding. Unless more corn can be sold, or more capital obtained, no additional water can be utilized.

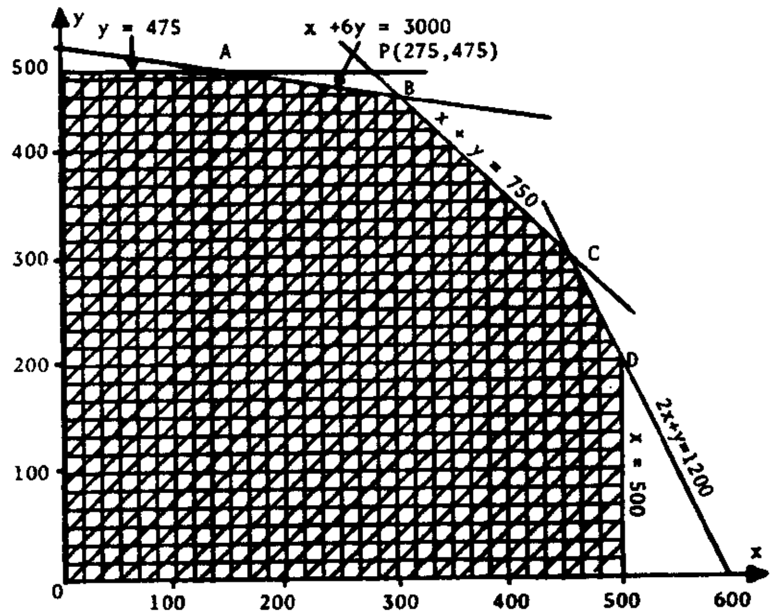


Figure 2. The intersection of the corn and capital constraints at P.

Example 2. The bakery in Example 1, page 5 of this module, finds that it can obtain unlimited amounts of additional custard. How many ounces should be ordered?

Step 1: Identify the two constraints that the custard constraint intersects. The custard constraint intersects the labor constraint at point B and the x-axis ($y \geq 0$) at point C.

Step 2: Solve these two constraints simultaneously to find their point of intersection. The simultaneous solution of $(1/8)x + (1/4)y = 10$ and $y = 0$ is $x = 80$, $y = 0$.

Step 3: Substitute the coordinates of this point in the left-hand side of the custard constraint. The result is the upper limit on the amount of custard that can be utilized.

$3(80) + 0 = 240$. Since no more than 240 ounces of custard can be utilized, the bakery should order only 120 in addition to the 120 it already has.

Exercise 4. The dry-cleaning company in Exercise 1 now finds that it can obtain an unlimited number of additional machines. How many should it order?

Exercise 5. The Broadway producers in Exercise 2 have decided to lower their original goal of reaching 2100 upper-income families. What new goal should they set for themselves?

Exercise 6. The farmer in Exercise 3 plans to increase his capital by taking a loan. How much money should he borrow?

2.5 Ranges for Objective Function Coefficients

So far we have considered the ways in which changes in the constraints affect the solution to a linear programming problem. We will now consider similar changes in the objective function. Suppose, for example, that the price of tomatoes went up, and as a result the farmer in Section 2.1 found that he could make a profit of \$4 on every bushel. How would this new circumstance affect the solution to the problem?

With the profit on a bushel of tomatoes raised from \$2 to \$4, the objective function becomes $P = 4x + 3y$. The values of this new objective function at the four vertices are given below.

Coordinates of Vertex		Value of Objective Function at Vertex
A	(150, 475)	$P = (4)(150) + (3)(475) = 2025$
B	(300, 450)	$P = (4)(300) + (3)(450) = 2550$
C	(450, 300)	$P = (4)(450) + (3)(300) = 2700$
D	(500, 200)	$P = (4)(500) + (3)(200) = 1700$

As a result of the change in the price of a bushel of tomatoes, the vertex that maximizes the objective function has shifted from B to C. The farmer should now raise 450 bushels of tomatoes and 300 bushels of corn, for a profit of \$2700.

When the profit on a bushel of tomatoes was \$2, the slope of the objective function was -2/3. When this profit increased to \$4, the slope of the objective function became -4/3. It was this change in the slope of the objective function that caused the optimal solution to shift from point B to point C.

B is the point of intersection of the water and capital constraints. For B to be the optimal solution, the objective function must lie on opposite sides of these two constraints when it passes through B, so that B is the only point of the shaded region to lie on the objective function. (See Figure 3.) This means that the slope of the objective function must lie between the slopes of the water and capital constraints. The slope of the capital constraint is -1; the slope of the water constraint is -1/6. B will be the optimal solution only if m, the slope of the objective function, lies between these two values, that is, only if $-1 < m < -1/6$.

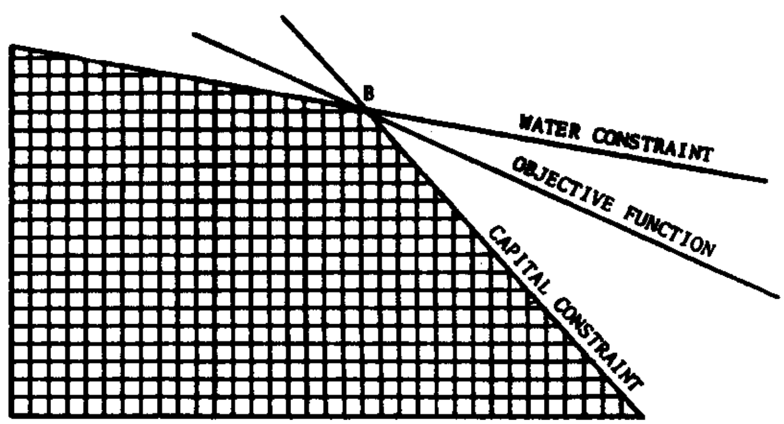


Figure 3. When B is the optimal solution, it is the only point in the shaded region to lie on the objective function.

(Since this is a maximization problem, the shaded area will lie entirely below the objective function when it passes through B; in a minimization problem the shaded area lies above the objective function at the optimal solution.)

Point C is the intersection of the capital and acreage constraints, whose slopes are -1 and -2, respectively. For C to be the optimal solution, it must be true that $-2 < m < -1$, where m is again the slope of the objective function. (If the slope of the objective function were -1, then B, C, and all the points between them would be optimal solutions.)

The slope of the objective function is $-t/c$, where t is the profit on a bushel of tomatoes and c is the profit on a bushel of corn. If c is fixed at \$3, t must satisfy the inequality $-1 < -t/3 < -1/6$ if the optimal solution is to remain at point B. If $-1 < -t/3$, then $1 > t/3$ and $3 > t$. If $-t/3 < -1/6$, then $t/3 > 1/6$ and $t > 1/2$. Thus, $\frac{1}{2} < t < 3$, and if the profit on a bushel of corn is fixed at \$3, B will be the optimal solution as long as the profit on a bushel of tomatoes lies between \$.50 and \$3. (Any change in the coefficients of the objective function will bring about changes in the shadow prices and their right-hand side ranges, whether or not the optimal solution changes.)

Example 3. The farmer in Section 2.1 finds that fluctuations in the market have caused the profit on a bushel of corn to fall to \$2 and the profit on a bushel of tomatoes to rise to \$2.50. Under these circumstances, what is the optimal solution? If the profit on a bushel of tomatoes remains unchanged at \$2.50, between what limits can the profit on a bushel of corn range without producing a change in the optimal solution?

Step 1: Determine the slope of the objective function.
The equation of the objective function is now $P = 2.50x + 2y$. Its slope is $-2.50/2$ or -1.25.

Step 2: Determine the vertex at which the objective function lies on opposite sides of the intersecting constraints, with the shaded area below it. Since $-2 < -1.25 < -1$, point C is the optimal solution.

Step 3: Determine the values of c for which $-2 < -2.50/c < -1$. These values define the range in the profit on a bushel of corn over which the slope of the objective function will remain between the slopes of the acreage and capital constraints when the profit on a bushel of tomatoes is \$2.50.

$-2 < -2.50/c$	$-2.50/c < -1$
$2 > 2.50/c$	$2.50/c > 1$
$c > 2.50/2 = 1.25$	$2.50 > c$

As long as the price of a bushel of corn is greater than \$1.25 but less than \$2.50, the slope of the objective function will lie between -2 and -1, and point C will be the optimal solution.

Exercise 7. The Broadway producers in Exercise 2 now find that bus advertisements have risen in price to \$4500. Will this change the optimal solution? If the cost of radio advertisements is still \$1000, for what range in the price of bus advertisements will point B be the optimal solution?

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3. CONCLUSION

Sensitivity analysis is a critically important part of linear programming. Because the parameters of a problem are subject to change, it is essential that shadow prices, right-hand side ranges, and ranges for objective function coefficients be determined. In a multidimensional linear program, however, the graphical methods employed in this module are not applicable. Instead, the simplex method, a complicated algorithm usually performed by a computer, yeilds this information automatically in the course of solving the problem. The principles involved, however, are much the same as those discussed here.

4. SAMPLE EXAM

Note: The problems in this sample exam are continuations of the problems in the sample exam in Linear Programming in Two Dimensions: I.

1. a) The watch company (problem 1, sample exam, p. 15, L.P. I) has been offered 25 additional quartz assemblies at $4 each. Explain why this offer should or should not be accepted.

   b) Competition forces the company to lower its prices, and the profit on each quartz watch falls to $10. Does this change the optimal solution? If so, how many of each type of watch should the company now manufacture to maximize its profits?

2. a) The dogfood manufacturers (problem 2, sample exam, p. 15, L.P. I) find that they can replace the calcium in the horsemeat and beef mixture with a calcium additive that they can buy for $.50/gm. Would it pay them to do this? If so, how many grams of the additive should they buy?

   b) If the price of beef remains fixed at $2 per pound, over what range can the price of horsemeat vary without changing the optimal solution?