Homework 11 Solutions

1. **Find whole numbers x and y such that $176x + 84y = 6$.**

   Using the Euclidean algorithm gives

$$\begin{aligned} 176 =& 2 \cdot 84 + 8 \\ 84 =& 10 \cdot 8 + 4 \\ 8 =& 2 \cdot 4. \end{aligned}$$

   Thus, $\gcd(176, 84) = 4$. Since 6 is not a multiple of 4, we see that $\boxed{\text{there is no solution}}$.

2. **Find whole numbers x and y so that $51x + 147y = 6$ *and* such that x is positive.**
   Using the Euclidean algorithm gives

$$\begin{aligned} 147 =& 2 \cdot 51 + 45 \\ 51 =& 45 + 6 \\ 45 =& 7 \cdot 6 + 3 \\ 6 =& 2 \cdot 3. \end{aligned}$$

   Thus we can run it backwards to express 3 as a combination of 51 and 147, and then multiply everything by 2 to get 6 as a combination of 51 and 147. Running the Euclidean algorithm backwards gives

$$\begin{aligned} 3 =& 45 - 7 \cdot 6 \\ =& 45 - 7(51 - 45) = -7 \cdot 51 + 8 \cdot 45 \\ =& -7 \cdot 51 + 8(147 - 2 \cdot 51) = 8 \cdot 147 - 23 \cdot 51. \end{aligned}$$

   and thus multiplying the entire equation by 2 gives $6 = 16 \cdot 147 - 46 \cdot 51$. We would be done, except that the question wants a solution with the coefficient of 51 positive, and our current coefficient of -46 doesn't qualify. To correct this, we add to our combination the equation $0 = -51 \cdot 147 + 147 \cdot 51$. Doing so gives $6 = -35 \cdot 147 + 101 \cdot 51$, and thus $\boxed{x = 101,\ y = -35}$ is a solution with the desired property.

   Alternatively, we could observe that the second line of the Euclidean algorithm already allows us to express 6 as a combination of 51 and 45. Running it backwards from there gives

$$\begin{aligned} 6 =& 51 - 45 \\ =& 51 - (147 - 2 \cdot 51) = -147 + 3 \cdot 51. \end{aligned}$$

   which immediately shows that $\boxed{x = 3,\ y = -1}$ is a solution (with x positive). Note also that is is a much "smaller" solution than above (in the sense of the absolute values of x and y). How could we have adjusted the coefficients above in order to find this solution?

3. **Describe how you could measure 2 cups of water given that you have three measuring cups, one for 12 cups, one for 20 cups, and one for 30 cups.**

   It's easy to see that no pair of these numbers has greatest common divisor of 2 or 1, and therefore we'll need to use all three cups (if we can do it at all). We can begin by using the Euclidean algorithm on 20 and 30:

$$\begin{aligned} 30 =& 20 - 10 \\ 20 =& 2 \cdot 10. \end{aligned}$$

Running it backwards shows that $10 = 30 - 20$. So using the 20 and 30 cup measuring cups is as good as having a 10 cup measuring cup. No we ask what can be done with our improvised 10 cup measuring cup and the 12 cup measuring cup. The Euclidean algorithm gives

$$\begin{aligned} 12 =& 10 + 2 \\ 10 =& 5 \cdot 2. \end{aligned}$$

We can run it backwards in order to get $2 = 12 - 10$. Recalling that our 10 is "really" $30 - 20$, we get $2 = 12 + 20 - 30$. So in order the measure 2 cups, we first measure out $12 + 20 = 32$ cups, and then take away 30, leaving us with the desired 2.