Homework 14 Solutions

1. (a) **Is $\binom{19}{7}$ divisible by 13?**

   We know that
   $$\binom{19}{7} = \frac{19!}{12! \cdot 7!}.$$
   We see that 13 divides the numerator but not the denominator, and thus the answer is $\boxed{\text{yes}}$, $\binom{19}{7}$ is divisible by 13.

   (b) **Is $\binom{19}{7}$ divisible by 10?**

   Because 10 is composite, we must reason a little more carefully here. Starting from the formula above, we can factor $\binom{19}{7}$ as follows:
   $$\begin{aligned}\binom{19}{7} &= \frac{19!}{12! \cdot 7!} \\ &= \frac{19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} \\ &= 2^2 \cdot 3 \cdot 13 \cdot 17 \cdot 19.\end{aligned}$$
   Since $10 = 2 \cdot 5$ and this prime factorization doesn't contain 5, we see that, $\boxed{\text{no}}$, $\binom{19}{7}$ is not divisible by 10.

   (c) **Is $\binom{19}{7}$ divisible by 6?**

   We know that $6 = 2 \cdot 3$. Referring to the above prime factorization of $\binom{19}{7}$, we see that it contains both 2 and 3; thus the answer is $\boxed{\text{yes}}$, $\binom{19}{7}$ is divisible by 6.

2. **Find the gcd of 18000 and 10935 by factoring these numbers.**

   Factoring gives us that
   $$\begin{aligned}18000 &= 18 \cdot 10^3 = 2 \cdot 3^2 \cdot 2^3 \cdot 5^3 \\ &= 2^4 \cdot 3^2 \cdot 5^3 \\ &\text{and} \\ 10935 &= 5 \cdot 2187 = 5 \cdot 3 \cdot 729 = 5 \cdot 3^2 \cdot 243 = 5 \cdot 3^3 \cdot 81 \\ &= 5 \cdot 3^4 \cdot 27 = 5 \cdot 3^5 \cdot 9 = 5 \cdot 3^7.\end{aligned}$$
   Comparing exponents, we see that the greatest common divisor is $3^2 \cdot 5$, or $\boxed{45}$.

3. **Recall the prime factorizations from the last problem.**

   (a) **How many positive whole numbers are divisors of *both* 18000 and 10935?**

   The common divisors of 18000 and 10935 are precisely the divisors of their gcd, 45. Since the prime factorization of 45 is $3^2 \cdot 5$, we see that we can specify all of its divisors by first making 1 of 3 choices for the exponent of 3 (either 0,1 or 2), and then 1 of 2 choices for the exponent of 5 (either 0 or 1). There are $3 \cdot 2 = 6$ ways to do this, and thus there are $\boxed{6}$ numbers that divide both 18000 and 10935.

   (b) **How many positive whole numbers are divisors of *either* 18000 or 10935? (Note that this includes numbers that are divisors of both.)**

   We know that $18000 = 2^4 \cdot 3^2 \cdot 5^3$, so it has $5 \cdot 3 \cdot 4 = 60$ divisors, by the same reasoning as above. Also, $10935 = 5 \cdot 3^7$, so it has $2 \cdot 8 = 16$ divisors. If we simply add these two numbers, we will double count the numbers that divide both 18000 and 10935. To correct for this, we must also subtract the number of numbers that divide both, which we know from the last part to be 6. Thus the number of positive integers that are divisors of either 18000 or 10935 is $60 + 16 - 6 = \boxed{70}$.