Homework 15 Solutions

1. **Let $m = 2^5 \cdot 3^7 \cdot 5^2 \cdot 13 \cdot 41$, and let $n = 2^3 \cdot 3 \cdot 5^5 \cdot 11 \cdot 13$.**

   (a) **What is the least common multiple of m and n? (Please leave your answer in factored form.)**

   For each prime in the factorizations of m and n, we take the larger of the two exponents. In this way, we find that the least common multiple is $\boxed{2^5 \cdot 3^7 \cdot 5^5 \cdot 11 \cdot 13 \cdot 41}$.

   (b) **How many positive whole numbers divide both m and n?**

   The numbers that divide both m and n are just the divisors of their gcd. To find their gcd, we take the smaller of the two exponents for each prime in the factorizations of m and n, which gives $\gcd(m, n) = 2^3 \cdot 3 \cdot 5^2 \cdot 13$. Counting the number of divisors of the gcd, we find that there are $4 \cdot 2 \cdot 3 \cdot 2 = \boxed{48}$ common divisors.

   (c) **How many positive whole numbers divide m but not n?**

   From its factorization, we see that m has $6 \cdot 8 \cdot 3 \cdot 2 \cdot 2 = 576$ divisors. By the last part, 48 of these also divide n. We conclude that there are $576 - 48 = \boxed{528}$ positive whole numbers that divide m but not n.

2. (a) **Is $\binom{22}{12}$ divisible by 23?**

   Recall that

   $$\binom{22}{12} = \frac{22!}{12!10!}.$$

   Now, 23 is prime and appears in neither the numerator nor the denominator. So, $\boxed{\text{no}}$, $\binom{22}{12}$ is not divisible by 23.

   (b) **Is $\binom{22}{12}$ divisible by 19?**

   Consulting our formula for $\binom{22}{12}$, we see that 19 appears in the numerator but not the denominator. Since 19 is prime, this implies that $\boxed{\text{yes}}$, $\binom{22}{12}$ is divisible by 19.

   (c) **Is $\binom{22}{12}$ divisible by 15?**

   Being divisible by 15 means being divisible by 3 and by 5. So lets check divisibility by these numbers. Since 5 is a bit easier, well do it first. There are four 5s appearing in the prime factorization of 22! one each contributed by 5, 10, 15, and 20. There are two 5s in 12! (coming from 5 and 10) and also two 5s in 10! (from 5 and 10 again). Thus there are four 5s in the numerator and four in the denominator; they all cancel out, meaning that $\binom{22}{12}$ is not divisible by 5. And since its not divisible by 5, it's not divisble by 15, so the answer is $\boxed{\text{no}}$.

   (d) **Find the greatest common divisor of $\binom{22}{12}$ and 35.**

   First we compute that $35 = 5 \cdot 7$. So, to find the gcd we must decide whether or not $\binom{22}{12}$ is divisible by 5 and 7. From the last part, we already know that its not divisible by 5. It remains to check divisibility by 7. Proceeding as above, there are three 7s in the numerator (coming from 7, 14, and 21) and two in the denominator (one in 12! and one in 10!). So, we find that $\binom{22}{12}$ is divisible by 7. It follows that the greatest common divisor is $\boxed{7}$.

3. For which k between 1 and 7, inclusive, is $\binom{8}{k}$ divisible by 8?

We compute that

$$\begin{aligned}
\binom{8}{1} = \binom{8}{7} &= 8, \\
\binom{8}{2} = \binom{8}{6} &= \frac{8 \cdot 7}{2} = 4 \cdot 7, \\
\binom{8}{3} = \binom{8}{5} &= \frac{8 \cdot 7 \cdot 6}{3 \cdot 2} = 8 \cdot 7, \quad \text{and} \\
\binom{8}{4} &= \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2} = 2 \cdot 7 \cdot 5.
\end{aligned}$$

Thus we see that the k between 1 and 7 such that $\binom{8}{k}$ is divisible by 8 are exactly $\boxed{1,\ 3,\ 5,\ \text{and } 7}$.