Homework 20 Solutions

1. **What is the last digit of 7^{991}?**

   This is the same as asking us to compute $7^{991} \pmod{10}$. Since 10 isn't prime, we can't use Fermat's theorem. Nonetheless, we can use successive squaring, which gives

$$\begin{aligned} 7^2 &\equiv 49 \equiv 9 \pmod{10}, \\ 7^4 &\equiv 81 \equiv 1 \pmod{10}. \end{aligned}$$

   Since $7^4 \equiv 1 \pmod{10}$, we have

$$7^{991} \equiv \left(7^4\right)^{247} \cdot 7^3 \equiv 7^2 \cdot 7 \equiv 9 \cdot 7 \equiv 63 \equiv \boxed{3} \pmod{10}.$$

2. (a) **Create a power table for arithmetic $\pmod{13}$. This will be a table whose rows correspond to numbers in arithmetic $\pmod{13}$ (that is, the numbers $\{0, 1, 2, \ldots, 12\}$), and whose entries are their various powers. Compute the powers from the 1st up to the 13th power for each number. (Recall the computational tricks from lecture; it's much less work that way.)**

   The most useful "trick" to recall is that the numbers larger than 6 can be re-written as negatives; for exmaple, $10 \equiv -3 \pmod{13}$. Thus, even powers of 10 are just the same as the corresponding power of 3, and odd powers are just the negative of the corresponding power of 3. This makes the bottom half of the table easy once you've done the top half. At any rate, the table looks like

	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	4	8	3	6	12	11	9	5	10	7	1	2
3	3	9	1	3	9	1	3	9	1	3	9	1	3
4	4	3	12	9	10	1	4	3	12	9	10	1	4
5	5	12	8	1	5	12	8	1	5	12	8	1	5
6	6	10	8	9	2	12	7	3	5	4	11	1	6
7	7	10	5	9	11	12	6	3	8	4	2	1	7
8	8	12	5	1	8	12	5	1	8	12	5	1	8
9	9	3	1	9	3	1	9	3	1	9	3	1	9
10	10	9	12	3	4	1	10	9	12	3	4	1	10
11	11	4	5	3	7	12	2	9	8	10	6	1	11
12	12	1	12	1	12	1	12	1	12	1	12	1	12

   (b) **Compute $2^{742} \pmod{13}$.**

   We see from the table (or Fermat's theorem), that $2^{12} \equiv 1 \pmod{13}$. Thus,

$$2^{742} \equiv \left(2^{12}\right)^{61} \cdot 2^{10} \equiv \boxed{10} \pmod{13},$$

   where we evaluated 2^{10} using the table.

   (c) **What is the 5th root of 4 $\pmod{13}$?**

   Looking down the x^4 column of the table, we see that 4 appears exactly once, in the 10 row. Hence, $4^{1/5} \equiv \boxed{10} \pmod{13}$.

   (d) **What is the 11th root of 9 $\pmod{13}$?**

   Looking down the x^{11} column of the table, we see that 9 appears exactly once, in the 3 row. Hence, $9^{1/11} \equiv \boxed{3} \pmod{13}$.