Homework 1 Solutions

Problems

1. **On a certain exam with ten questions, a student must answer 7 and omit 3. How many ways can he or she make the choice? Later, the student notices that problem #1 is easy. After doing that problem, how many ways are there to finish the exam?**

   The student must choose 7 questions from 10 to answer; the number of ways of doing this is $\boxed{\binom{10}{7}}$. Of course, the student could choose which 3 to omit; an equivalent form of the answer is $\binom{10}{3}$.

   After answering problem #1, the student must choose 6 of the remaining 9 questions; there are $\boxed{\binom{9}{6}}$ ways to do this (or equivalently $\binom{9}{3}$).

2. **In the small town of New Baden, Illinois, the town council is made up of 10 teachers, 5 firefighters, and 5 police officers. How many ways are there to choose a committee of 7? What if there must be exactly one teacher on the committee? How about at least one teacher on the committee?**

   There are $\boxed{\binom{20}{7}}$ ways to choose a committe of 7 from 20.

   If there must be exactly one teacher on the committee, there are 10 ways to choose this teacher, and $\binom{10}{6}$ ways to choose the other 6 people from the 10 non-teachers. By the multiplication principle, the number of committees with exactly one teacher is $\boxed{10 \cdot \binom{10}{6}}$.

   Requiring at least one teacher is the same as excluding committees made up entirely of non-teachers. There are $\binom{10}{7}$ committees with only police officers and firefighters. Subtracting this from the total gives $\boxed{\binom{20}{7} - \binom{10}{7}}$.

   Common Mistake: Some students answered this problem by saying that there are 10 choices for the teacher that the committee must have, and then the remaining 6 members can be chosen freely from among the 19 other people, for a total of $10 \cdot \binom{19}{6}$. However, this must be wrong since $10 \cdot \binom{19}{6} = 271320$, which is significantly more than the total number of committees, $\binom{20}{7} = 77520$. This technique overcounts the committees with more than one teacher, since any of the teachers on such a committee could be designated the special teacher who is chosen first.

3. **It's time to look ahead to next season for the Chicago Cubs. Suppose they play 162 games, and suppose that they end up winning 88 of those games. How many possible win/loss records are there? By a win/loss record, I mean a list of which games were won and which were lost. What if we drop the condition that the Cubs win 88 games, and only require that they win more than half of their games?**

   We must simply choose which 88 of the 162 games the Cubs won; the rest of the games will be losses. There are $\boxed{\binom{162}{88}}$ ways to make this choice.

   For the second part, we must add up a number of different cases: to win more than half of their games, the Cubs must win at least 82 times. So they could have 82 wins, 83 wins, 84 wins, etc., all the way up to 162 wins. Following the first part, the number of ways to make each of these choices is, respectively, $\binom{162}{82}, \binom{162}{83}, \ldots, \binom{162}{162}$. Since these are all mutually exclusive possibilities for winning seasons, we can just add them up to find the total number of winning records:

$$\binom{162}{82} + \binom{162}{83} + \cdots + \binom{162}{162}$$

This gives the correct answer, although it involves summing 81 terms.

As alluded to in the hint, there is an approach which gives the answer in a much more manageable and concise form. Notice that a record is either winning, losing, or a tie; or expressed as an equation,

$$\#\text{ total records} = \#\text{ with less than 81 wins} + \#\text{ with 81 wins} + \#\text{ with more than 81 wins.}$$

Further, the number of records with less than 81 wins is the same as the number of records with more than 81 wins (think about this for a second if it's not immediately clear). The total number of records is 2^{162}, because each of 162 games can be either a win or a loss. The number of records with exactly 81 wins is $\binom{162}{81}$. If we let W denote the number of winning records (and also the number of losing records), the above equation becomes

$$2^{162} = W + \binom{162}{81} + W.$$

Solving for W gives

$$\boxed{W = \frac{2^{162} - \binom{162}{81}}{2}}.$$