Homework 5 Solutions

1. **How many distinct ways are there to rearrange the letters in "MISSISSIPPI"? How about "WOOLLOOMOOLOO" (a town near Sydney, Australia)?**

   We first choose 4 of the 11 spaces in which to put the I's; there are $\binom{11}{4}$ ways to do this. Next, choose 4 of the remaining 7 spaces in which to put the S's; there are $\binom{7}{4}$ such choices. Finally, choose 2 of the remaining 3 spaces in which to put the P's; there are $\binom{3}{2}$ ways to do this. The M must then go in the sole remaining space. By the multiplication principle, the total number of choices is $\binom{11}{7}\binom{7}{4}\binom{3}{2}$. An equivalent way of expressing this answer is using the multinomial coefficient $\boxed{\binom{11}{4,4,2,1}}$; they are both equal to $\frac{11!}{4!4!2!}$.

   Similarly, "WOOLLOOMOOLOO" contains 8 O's, 3 L's, a W and an M. The number of ways to rearrange its letters is $\binom{13}{8}\binom{5}{3}\binom{2}{1} = \boxed{\binom{13}{8,3,1,1}} = \frac{13!}{8!3!}$.

2. **At a certain small boarding school, the incoming class of 12 needs to be assigned to dormitory rooms. There is one 5-person room, a quad, a double, and a single. In how many ways can the assignment be made?**

   We first choose 5 students to put into the quintuple room; we can make this choice in $\binom{12}{5}$ ways. Next choose 4 from the remaining 7 for the quad; there are $\binom{7}{4}$ ways to do this. Finally, choose 2 of the remaining 3 to share the double; there are $\binom{3}{2}$ such choices. The last person will be assigned to the single. The total number of choices is $\binom{12}{5}\binom{7}{4}\binom{3}{2}$, or $\boxed{\binom{12}{5,4,2,1}} = \frac{12!}{5!4!2!}$.

3. **You roll two standard 6-sided dice. What is the probability that their sum is at least 8? What is the probability that their difference is at least 2?**

   If an "outcome" is a sequence of two numbers between 1 and 6, then there are $6 \cdot 6 = 36$ possible outcomes, all of which are equally likely.

   To find the number of outcomes which have a sum of at least 8, we need to count the number of sequences that sum to 8, plus the number that sum to 9, and so on, up to 12. These are:

   - a 2 and a 6, or a 3 and a 5, or a 4 and a 4, or a 5 and a 3, or a 6 and a 2; or
   - a 3 and a 6, or a 4 and a 5, or a 5 and a 4, or a 6 and a 3; or
   - a 4 and a 6, or a 5 and a 5, or a 6 and a 4; or
   - a 5 and a 6, or a 6 and a 5; or
   - a 6 and a 6.

   This makes 15 total possibilities. (You can avoid listing them all by referring back to the totals on page 72 of the sourcebook. There are five ways to get an 8, four ways to get a 9, and so on, so there are $5+4+3+2+1 = 15$ ways in total.) Hence, the probability of getting a sum of at least 8 is $\frac{15}{36} = \boxed{\frac{5}{12}}$.

   To find the number of outcomes which have a difference of at least 2, it is slightly easier to count the outcomes which *do not* have a difference of at least 2; that is, those which *have a difference of 0 or 1*. These consist of: "doubles" outcomes, where the difference is 0 (six possibilities); outcomes where the first roll is greater than the second by 1 (five possibilities); and outcomes where the second roll is greater than the first by 1 (five possibilities).

   This makes 16 total possibilities, so there are $36 - 16 = 20$ ways of getting a difference of *at least* 2. Hence, the probability of getting a difference of at least 2 is $\frac{20}{36} = \boxed{\frac{5}{9}}$.

4. **What is the probability that when four cards are dealt at random from an ordinary deck of 52, each suit is present?**

   If an "outcome" is a hand of four cards dealt from a deck of 52, then there are $\binom{52}{4}$ possible outcomes. A *favorable* outcome is a hand in which the four cards have all different suits. Like we did in lecture for poker hands, we can count the number of favorable outcomes by counting the number of ways to *specify* such a hand: here, we just have to specify the denomination of the spade card, the denomination of the heart card, the denomination of the diamond card, and the denomination of the club card. There are 13 options for each choice of denomination, and each choice can be made independently of the others, so there are $13 \cdot 13 \cdot 13 \cdot 13 = 13^4$ total favorable outcomes. Thus, the probability is $\boxed{\frac{13^4}{\binom{52}{4}}}$.

   Alternative method: If you wish, it's possible to think of an "outcome" as a *sequence* of four dealt cards — there are $52 \cdot 51 \cdot 50 \cdot 49$ of these. Then, the number of such sequences where each suit is present can be computed by counting the number of allowable possibilities for a card at each successive deal: there are 52 possibilities for the first card, but only 39 for the second card (since we wish the suit to be different), and only 26 for the third card (since we again wish the suit to be different from the previous two) and only 13 for the fourth (same reasoning). So there are $52 \cdot 39 \cdot 26 \cdot 13$ sequences in all, making a probability of $\boxed{\frac{52 \cdot 39 \cdot 26 \cdot 13}{52 \cdot 51 \cdot 50 \cdot 49}}$, which is the same as above. (**Warning**: this method can be clumsy if you try using it to count other kinds of card hands. It works nicely here because the number of favorable options for each successive dealt card is *independent* of the previously dealt cards.)