

Homework 5 Solutions

1. **In the game of Yahtzee, the players roll 5 dice at a time, and the best possible roll is a “yahtzee,” which is when all 5 dice come up the same. What is the probability of rolling a yahtzee in one roll? Another valuable roll is the “large straight,” where the dice are all in a row (either 1,2,3,4,5 or 2,3,4,5,6). What is the probability of rolling a large straight in one roll?**

The total number of (equally likely) outcomes when rolling 5 dice is 6^5 , since there are 6 choices for each of the dice. The number of ways of having 5 dice all showing the same number is 6, 1 for each of the 6 possible numbers (for example, there is only one way of having all the dice show 1, namely, 1,1,1,1,1, and similarly for the other numbers). Thus, the probability of rolling a “yahtzee” is $\boxed{6/6^5 = 1/6^4}$.

Next we need to count the number of ways of having a large straight. As the problem mentions, there are two possible “templates,” 1,2,3,4,5 or 2,3,4,5,6. We begin by counting the number of ways to roll a 1,2,3,4,5. Since the numbers can appear in any order, we see that we just need to choose an order for these 5 digits. But we know that the number of ways of putting 5 different things in order is $5!$. Hence there are $5!$ ways to get a large straight using the numbers 1,2,3,4,5. The same reasoning shows that there are $5!$ ways of getting a large straight using the numbers 2,3,4,5,6. Adding these, we see that there are $2 \cdot 5!$ ways to get a large straight, and thus the probability of rolling one is $\boxed{2 \cdot 5!/6^5}$.

2. **At the annual meeting of the National Elkhound Association, there are 8 propositions on the ballot. Each member may vote “yes,” “no,” or “abstain” on each proposition, but each ballot can have no more than one abstention. In how many ways can a ballot be cast?**

There are two possibilities: a voter may abstain not at all, or just once. In the former case, the voter has a choice of “yes” or “no” on each of eight questions, for a total of 2^8 choices. In the latter case, the voter may choose to abstain on any of the 8 questions, and then has a choice of “yes” or “no” on each of the remaining 7; there are thus $8 \cdot 2^7$ such ballots. The total number of legal ballots is their sum, or $\boxed{2^8 + 8 \cdot 2^7}$.

3. **How many distinct rearrangements of “MISSISSIPPI” are possible if we don’t allow the I’s to be all together? What if we don’t allow the I’s to be all together or the S’s to be all together?**

We will use the subtraction principle. In particular, the number of rearrangements without all the I’s together is equal to the total number of rearrangements minus the number in which the I’s are all together. The total number of rearrangements is $\binom{11}{4,4,2,1}$, since there are 11 spots total in which we wish to place four S’s, four I’s, two P’s, and one M. To compute the number of rearrangements with all the I’s together, we will treat the I’s as a single block (if we think of the letters as being on Scrabble tiles, we’re gluing the four I tiles together). Thus we now have 8 spots total, which we wish to fill with four S’s, two P’s, one M, and one block of I’s, and there are $\binom{8}{4,2,1,1}$ ways to do so. Thus the number of rearrangements where we don’t allow all

the I’s to be together is $\boxed{\binom{11}{4,4,2,1} - \binom{8}{4,2,1,1}}$, or, written in terms of factorials, $\boxed{\frac{11!}{4!4!2!} - \frac{8!}{4!2!}}$.

For the next part, we will again use the subtraction principle, although in this case there is an extra twist. In particular, the number of rearrangements in which we don’t allow all the I’s together or all the S’s together is equal to the total number of rearrangements minus the number in which the I’s are all together, minus the number in which the S’s are all together, *plus* the number in which the I’s are together and the S’s are together. This is because we need to get rid of the ones with the I’s together and also the ones with the S’s together, but if

we just subtract those two quantities we've double-counted the ones with the I's together and the S's together. To correct this overcount, we add back in the number in which the I's are together and the S's are together. Once we have this figured out, we can count the number of arrangements corresponding to each term just as before. The total number of arrangements and the number with the I's together are the same as above. The number with the S's together is also $\binom{8}{4,2,1,1}$, since if we treat them as one block we have eight slots for four I's, two P's, one M, and one block of S's. To count the number with the I's together and the S's together, we treat each of them as a single block. Thus the number is $\binom{5}{2,1,1,1}$, since we have five slots for two P's, one M, one block of I's, and one block of S's. Putting this all together, we see that the number of rearrangements in which we don't allow all of the I's together or all of the S's together is

$$\boxed{\binom{11}{4,4,2,1} - 2\binom{8}{4,2,1,1} + \binom{5}{2,1,1,1}},$$

or, written in terms of factorials,

$$\boxed{\frac{11!}{4!4!2!} - 2\frac{8!}{4!2!} + \frac{5!}{2!}}.$$