Homework 7 Solutions

1. **In walking from work back home at the end of the day, Grigor must travel six blocks east and four blocks south. How many possible routes can he take, assuming that he goes either east or south at each intersection? How many routes are there if he needs to stop by the dry-cleaners on the way home, which is located at the center of the grid (that is, two blocks south and three east of his workplace)?**

   Grigor will travel a total of 10 blocks, of which any 6 can be going east. Once we pick the 6 blocks on which he will go east, we've completely specified his route, as he must go south on the other 4. Thus there are $\boxed{\binom{10}{6} = \binom{10}{4}}$ routes.

   For the second part, we split his route into two parts. First, he must go from his work to the dry-cleaners. By the same logic as before, there are $\binom{5}{3}$ such routes. Then he must go from the dry-cleaners to his home; again there are $\binom{5}{3}$ ways for him to do so. Since there are $\binom{5}{3}$ ways to choose the first part of his journey and $\binom{5}{3}$ ways to choose the second part, the multiplication principle tells us that there are $\boxed{\binom{5}{3}^2 = \binom{5}{2}^2}$ total routes.

2. **Express $\binom{7}{4} + \binom{7}{5}$ as a single binomial coefficient.**

   Express $\binom{9}{4} + \binom{9}{3} + \binom{10}{3}$ as a single binomial coefficient.

   From lecture (or the book), we know that the binomial coefficients obey the relation

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

   (There are two ways we've discussed of thinking about this fact. One is to consider choosing objects from a collection in which you "mark" one of the objects as special. The other is to think about the relationship between binomial coefficients and Pascal's triangle.) If we let $n = 8$ and $k = 5$, we see that

$$\binom{7}{4} + \binom{7}{5} = \boxed{\binom{8}{5}}.$$

   For the second part, we first observe that

$$\binom{9}{4} + \binom{9}{3} = \binom{10}{4},$$

   by applying our formula with $n = 10$ and $k = 4$. Having done that, we now see that our formula gives

$$\binom{10}{4} + \binom{10}{3} = \binom{11}{4}$$

   Combining all of this, we have

$$\begin{aligned} \binom{9}{4} + \binom{9}{3} + \binom{10}{3} &= \binom{10}{4} + \binom{10}{3} \\ &= \boxed{\binom{11}{4}}. \end{aligned}$$

3. **Suppose you roll three fair dice. What is the probability that their sum is 5?**

   We can think of our outcomes as sequences of 3 numbers, each between 1 and 6. Each of these outcomes is equally likely, and the toal number of outcomes is 6^3. We now need to determine the number of favorable outcomes. In order to get a sum of 5, at least one of the die must be 1. Then we need the other two dice to sum to 4, which means that they must either be a 3 and a 1 or two 2's. Hence, to get a sum of 5 we need that our three die are either 1, 1, 3 or 1, 2, 2. Since they can appear in any order, we need to determine how many ways there are to roll 1, 1, 3 (in any order). But this is the same as asking how for how many ways you can rearrange those numbers, and we know that the answer is $\binom{3}{1} = 3$ (since you need to choose which spot the 3 goes in and then you're done). Similarly, there are 3 ways to roll 1, 2, 2 (in any order). Thus, in total, there are 6 favorable outcomes. Dividing this by the total number of outcomes, we see that the probability of rolling three dice that sum to 5 is

   $$\boxed{\frac{6}{6^3} = \frac{1}{6^2} = \frac{1}{36}}.$$

4. **Suppose that you choose three people at random. What is the probability that two of them were born on the same day of the week (assume that a randomly chosen person is equally likely to have been born on any day of the week)?**

   We interpret this to mean that at least two of them were born on the same day. An outcome is an assignment of a day of the week to each person. Thus there are 7^3 total outcomes, and the problem tells us that they are all equally likely. To find the number of favorable outcomes, we will use the subtraction principle. To find the number of outcomes with at least two people sharing a birthday, we must subtract the number of outcomes with all three birthdays different from the total number of outcomes. The number of outcomes with all three birthdays different is $7 \cdot 6 \cdot 5$, since we are assigning 7 days to 3 people, with no repeats allowed. Thus the number of favorable outcomes is $7^3 - 7 \cdot 6 \cdot 5$. Dividing this by the total number of outcomes, we see that the probability that at least two people will share a birthday is

   $$\boxed{\frac{7^3 - 7 \cdot 6 \cdot 5}{7^3} = 1 - \frac{30}{7^2} = \frac{19}{49}}.$$

   Note: Some people interpreted the question to mean that exactly two people should share a birthday. In this case, there are a couple of ways of solving the problem, but here we present the one most similar to the above. Having computed the number of outcomes with at least two people sharing a birthday, we need to subtract the number with all three sharing a birthday in order to get the number with exactly two people sharing a birthday. The number of outcomes with all three sharing a birthday is 7, since we simply pick a day of the week for them all to share. Hence the number of favorable outcomes is $7^3 - 7 \cdot 6 \cdot 5 - 7$, and the probability of exactly two people sharing the same birthday is

   $$\frac{7^3 - 7 \cdot 6 \cdot 5 - 7}{7^3} = \frac{18}{49}.$$

   For practice, you might try to find a simpler way of doing this directly.