Practice Midterm 1 Solutions

1. (a) **How many numbers are there between 1000 and 5999?**

   The famous formula gives it:

   $$5999 - 1000 + 1 = \boxed{5000}.$$

   (b) **Among the numbers of part (a) how many of them have none of their digits the same? (E.g., 4984 is not allowed but 4985 is OK.)**

   The number of possible digits for the "leftmost" digit is 5; this being chosen, the number for the next digit to be chosen is 9; the next 8, and the next 7. Grand total: $\boxed{5 \times 9 \times 8 \times 7}$.

   (c) **Among the numbers of part (a) how many of them have exactly two of their digits the same?**

   Break this up into two cases:

   (1) where the repeated digit is the leftmost. So there are 5 choices for that repeated digit, and 3 choices for the placement of the leftmost repeated digit; there are then 9 choices for the first of the nonrepeated digits and 8 for the second. Grand total: $5 \times 3 \times 9 \times 8$.

   (2) where the repeated digit is not the leftmost. So there are then 5 choices for the leftmost digit and 3 choices for the placement of the two repeated digits; 9 choice for that repeated digit, and 8 choices for the nonleftmost, nonrepeated, digit. Grand total: $5 \times 3 \times 9 \times 8$.

   Thus, the answer is:

   $$\boxed{5 \times 3 \times 9 \times 8 + 5 \times 3 \times 9 \times 8 = 2160}.$$

   (d) **Among the numbers of part (a) how many of them are not divisible by 6?**

   Smallest number in the range divisible by 6 is $1002 = 6 \times 167$ and largest divisible by 6 is $6 \times 999 = 5994$. So, the number of numbers in the range divisible by 6 is $999 - 167 + 1 = 833$, and the answer to the question, then, is $\boxed{5000 - 833}$.

   (e) Among the numbers of part (a) how many of them are not divisible by 6 or by 9?

   Smallest divisible by 9 is $1008 = 9 \times 112$ and largest divisible by 9 is $9 \times 666 = 5994$. So, the number of numbers in the range divisible by 9 is $666 - 112 + 1 = 555$.

   Note, now that $18 = \text{lcm}(6, 9)$. Smallest divisible by 18 is $1008 = 18 \times 56$ and largest divisible by 18 is $18 \times 333 = 5994$. So, the number of numbers in the range divisible by 18 is $333 - 56 + 1 = 278$. Answer: Using the subtraction principle: $\boxed{5000 - 833 - 555 + 278}$.

2. (a) **How many numbers are there between 11 and 111?**

   Using the formula from class we get $\boxed{111 - 11 + 1 = 101}$.

   (b) **How many of them are even and not divisible by 3?**

   The even numbers between 11 and 111 are $12, 14, 16 \ldots 110$ which can be written as $2 \times 6, 2 \times 7, 2 \times 8, \ldots, 2 \times 55$. To find how many of those are not divisible by 3 it is enough to count the numbers between 6 and 55 which are not divisible 3. The ones divisible by 3 are $3 \times 2, 3 \times 3, \ldots, 3 \times 18$ and they are $18 - 2 + 1 = 17$ in total. The numbers between 6 and 55 are $55 - 6 + 1 = 50$ and using the subtraction principle the number of the ones not divisible by 3 is $\boxed{50 - 17 = 33}$.

   (c) **How many of them are odd and divisible by 3?**

   The numbers between 11 and 111 divisible by 3 are $3 \times 4, 3 \times 5, 3 \times 6, \ldots, 3 \times 37$ and there are $37 - 4 + 1 = 34$ of them. To find the odd ones among these is the same as finding the odd numbers between 4 and 37. They are $2 \times 2 + 1, 2 \times 3 + 1, 2 \times 4 + 1, \ldots, 2 \times 18 + 1$ and their number is $\boxed{18 - 2 + 1 = 17}$.

(d) **How many of them are divisible by neither 2, 3 or 37?**

There are three numbers between 11 and 111 which are divisible by 37 - $37, 74, 111$. That means that we are essentially trying to count the numbers between 11 and 110 , excluding 37, which are divisible by neither 2 or 3. It is enough to count the numbers between 11 and 110 which are divisible by neither 2 or 3 and then subtract 1 from the final answer (because of 37). We use the double subtraction principle:

- there are $55 - 6 + 1 = 50$ even numbers
- there are $36 - 4 + 1 = 33$ ones divisible by 3
- there are $18 - 2 + 1 = 17$ divisible by 6 (i.e by both 2 and 3)

The final answer is $\boxed{100 - 50 - 33 + 17 - 1 = 33}$.

3. **Please compute the following (that is, your answer should be a whole number).**

(a) $\binom{5}{3}$

We know that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. In our case this means that $\binom{5}{3} = 5!/(3!2!)$. The 3! cancels everything in the 5! except the 5 and the 4, so this simplifies to $5 \cdot 4/2!$. We know that $2! = 2$, so multiplying this out shows that $\binom{5}{3} = \boxed{10}$.

(b) $\binom{1,000,113}{1,000,112}$

Observe that $\binom{1,000,113}{1,000,112} = \binom{1,000,113}{1}$. If we ask how many ways there are to choose 1 item out of 1,000,113, it is clear that there are $\boxed{1,000,113}$ to do so. Note that you get the same answer if you write out the factorials as above; everything except 1,000,113 immediately cancels.

(c) $\binom{6}{4,2,1,1}$

We know that $\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1! k_2! \cdots k_m!}$. In the present case, this means that $\binom{6}{4,2,1,1} = 6!/(4!2!1!1!)$. Now $1! = 1$, $2! = 2$, and the 4! cancels everything in the 6! except for $6 \cdot 5$. Thus this simplifies to $6 \cdot 5/2 = \boxed{15}$.

(d) $\binom{4}{1,1,1,1}$

Again, we apply the above formula to see that $\binom{4}{1,1,1,1} = 4!/(1!1!1!1!)$. Since $1! = 1$, this is just 4!. (Note that this multinomial coefficient is the number of ways to arrange 4 objects if they are all distinct, so we shouldn't be surprised that it equals 4!.) Finally, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = \boxed{24}$.

4. (a) **Suppose that a license plate has seven spots, and that valid characters for these slots are the 26 letters (A through Z) and the 10 digits (0 through 9). How many possible license plates are there?**

There are 36 choices (26 letters plus 10 digits) for each of seven spots. Thus the multiplication principle tells us that the total number of choices is $\boxed{36^7}$.

(b) **Now suppose that you want your license plate to contain exactly 3 letters. How many possible plates are there?**

First, we need to choose which spots get letters and which get digits; we can think of this as determining the "template" for the solution. We can do this by choosing which 3 spots get letters (the rest will get digits); there are $\binom{7}{3}$ ways of doing this. After we've chosen where the letters go and where the digits go, we need to choose which letters and digits will be used. There are 26 choices for each of 3 letters, so there are 26^3 ways to choose the letters. Similarly, there are 10^4 ways to choose the 4 digits. Hence there are $26^3 \cdot 10^4$ ways to choose the particular letters and digits used. Since each of our $\binom{7}{3}$ templates gives $26^3 \cdot 10^4$ license plates, the total number of possible plates is $\boxed{\binom{7}{3} 26^3 \cdot 10^4}$.

(c) **Now suppose that, in addition to having exactly 3 letters, you also refuse to allow these letters to be all in a row. How many possible plates are there meeting these conditions?**

We will use the subtraction principle. We wish to take the number of plates with exactly 3 letters (which we know from the last part) and then subtract the number of such plates with all 3 letters in a row. Since we're now interested in the number of plates with all 3 letters together, we can think of these 3 letters as forming a single block. Thus we have 5 slots which we wish to fill with 4 digits and a block of letters. There are $\binom{5}{1} = 5$ ways of doing so, since we simply need to pick a slot for the letters. So there are 5 templates when we want all 3 letters together. Once we have our template, we need to pick specific digits and letters. Just as before, there are $26^3 \cdot 10^4$ ways to do this (since we still have 3 letters and 4 digits). We conclude that there are $5 \cdot 26^3 \cdot 10^4$ plates with all 3 letters together. Using this and the solution to the previous part in our subtraction principle, we see that the number of plates (with exactly 3 letters) such that all 3 letters aren't together is $\boxed{\binom{7}{3}26^3 \cdot 10^4 - 5 \cdot 26^3 \cdot 10^4 = \left(\binom{7}{3} - 5\right) 26^3 \cdot 10^4}$.

5. (a) **How many two-card hands are there out of a deck of 52 that consists of two cards of contiguous denominations (that is a hand that looks like {A,2}, or {2,3}, or {3,4} ,. . . , or {K,A}?**

The total number of two-card hands is $\binom{52}{2} = 52 \times 51/2 = 1326$. Of course, the total number of ordered two-card hands (i.e., where you stipulate which of the two cards is the first, and which the second) is twice that, i.e. $52 \times 51 = 2652$. Now if you first specify one of the cards of your two-card hand, you have 52 choices. Given that choice, there are $2 \times 4 = 8$ further choices for contiguous cards to make up your two-card hand with contiguous denominations, making a total of 8×52 (ordered) two-card hands with contiguous denominations, so our answer is half that, i.e. $\boxed{4 \times 52 = 208}$.

(b) **If you are dealt two cards at random, what is the probability that you have such a two-card hand?**

The probability is {# of favorable hands} over {total # of hands}. So $\boxed{208/1326}$. Another way of seeing this is to note that for any one of the cards in this hand, the probability that the other card forms a favorable two-card hand is 8/51 for evident reasons.

(c) **What is the probability of drawing a poker hand having exactly two pairs (recall that a pair is two cards of the same denomination), one of the pairs consisting of red suits (hearts, diamonds) and the other pair consisting of black suits (spades, clubs)?**

The number of "red" pairs is 13; the number of "black" pairs is 13 but you dont want your black pair to have the same denomination as your red pair so the total number of combos (red pair, black pair; of different denominations) is 13×12. Given such a red and black pair combo, the available further cards consists in all cards of a denomination different from the red and the black denomination. That is, it must be different from those 8 cards, leaving 44 cards good for forming a favorable hand. That gives us that the grand total of "favorable hands" is $13 \times 12 \times 44$. So the probability is

$$\boxed{\frac{13 \times 12 \times 44}{\binom{52}{5}}}.$$

6. (a) **You are dealt, at random, 4 cards from a standard 52 card deck. What is the probability that the denominations are all consecutive (that is, you have the 4-card version of a straight)? Here the Ace can be either high or low, so that $\{A, 2, 3, 4\}$ and $\{J, Q, K, A\}$ both count.**

We can think of our outcomes as a collection of 4 cards out of the 52; these are equally likely. Then there are $\binom{52}{4}$ total outcomes. Next we need to count the favorable outcomes.

We first choose the denominations in our "4-card straight." The lowest denomination can be anything from an Ace (low) up to a Jack (any higher and 4 cards won't fit). Thus there are 11 choices for the denomination of the low card. Once we choose the low card, the rest of the denominations are determined as well, since they must be consecutive. Once we've chosen our denominations, we must give each card a suit. Since there are 4 cards and each can be any of 4 denominations, there are 4^4 ways to assign suits to our cards. So by first choosing the denominations and then the suits, we see that there are $11 \cdot 4^4$ possible "4-card straights." Since these are our favorable outcomes, we divide by the total outcomes to find that the probability of being dealt a "4-card straight" is $\boxed{\frac{11 \cdot 4^4}{\binom{52}{4}}}$.

(b) **What is the probability that you have 2 pair, that is, that you have two cards of one denomination and two cards of a different denomination?**

Again, the total number of outcomes is $\binom{52}{4}$. We proceed to count the number of favorable outcomes, that is, the number of 2 pair hands. We start by choosing denominations for our cards. Out of the 13 denominations, we need 2 of them to be the denominations of our 2 pairs. Thus, there are $\binom{13}{2}$ ways of picking the denominations. Next we need to choose suits for our cards. For the first pair, there are 4 possible suits, of which we must choose 2. So there are $\binom{4}{2}$ ways to assign suits to the first pair. Similarly, there are $\binom{4}{2}$ ways to assign suits to the second pair, and thus there are $\binom{4}{2}^2$ ways to assign suits to all 4 cards. By first assigning denominations to our cards and then suits, we see that there are $\binom{13}{2}\binom{4}{2}^2$ possible 2 pair hands, and thus that many favorable outcomes as well. Dividing by the total number of outcomes, we see that the probability of being dealt 2 pair is $\boxed{\frac{\binom{13}{2}\binom{4}{2}^2}{\binom{52}{4}}}$.

7. (a) **How many different ("mathematical") words can be made by rearranging the letters of the word INSTINCTS?**

The letters of INSTINCTS comprise one C and two each of I, N, T, S. Hence the number of possible rearrangements is $\boxed{\binom{9}{1,2,2,2,2}} = \boxed{9!/(2!)^4}$.

(b) **By a palindrome we mean a ("mathematical") word that is the same when spelled backwards, like RACECAR. How many different palindromes can be made by rearranging the letters of the word INSTINCTS?**

In a palindromic rearrangement, the C must be middle letter. The first four letters are I, N, T, S ordered arbitrarily, which can be done in 4! ways. Once we have chosen this order, the final four letters must be the same in reverse order, so we have no further choices. The total number is thus $\boxed{4!}$ (numerically, 24).

8. (a) **How many six-letter words are there with more consonants than vowels, but at least one vowel?**

Such a word must contain either exactly one or exactly two vowels. If the word has exactly two vowels, we first choose the position of these vowels; there are $\binom{6}{2}$ ways to do this. Then we choose which vowels occur in these two spots; for each one there are 5 choices (A, E, I, O, U), hence a total of 5^2. For each of the remaining four spots we need to select one out of 21 consonants, independently of each other. There are 21^4 ways to do this. By the multiplication principle the number of six-letter words with exactly two vowels is $\binom{6}{2} \cdot 5^2 \cdot 21^4$.

In the same way we find that the number of six-letter words having exactly one vowel is $\binom{6}{1} \cdot 5 \cdot 21^5$.

Therefore the answer is $\boxed{\binom{6}{2} \cdot 5^2 \cdot 21^4 + \binom{6}{1} \cdot\ 5 \cdot 21^5}$.

(b) **What if instead we want to count words with exactly five consonants but no repeated letters?**

We start by choosing the position of the one vowel; there are $6 = \binom{6}{1}$ choices for this. Then we choose which vowel goes into this spot – there are 5 possibilities. Finally we have 5 remaining spots for the consonants. We have 21 choices for the first consonant, 20 choices for the second,..., and $17 = 21 - 5 + 1$ choices for the fifth; this makes $21 \cdot 20 \cdots 17 = 21!/16!$ choices. Applying the multiplication principle we obtain the final answer: $\boxed{6 \cdot 5 \cdot \frac{21!}{16!}}$.

(Assume that Y is counted as consonant.)

9. **In the game of Yahtzee five dice are thrown simultaneously. What's the probability...**

(a) **...to throw a pair and a triple (but not all five equal)?**

An outcome of throwing 5 dice is a sequence of five numbers from 1 to 6; so the total number of possible outcomes is 6^5. To find the number of favorable outcomes, i.e. those that form a pair and a triple, we first choose in which of the five spots the two equal numbers appears; there are $\binom{5}{2}$ ways to do that. The three equal numbers will then be in the remaining three spots. Now we choose which number appears twice; there are $6 = \binom{6}{1}$ choices for this. For the number that occurs three times any of the remaining five numbers among 1, 2, ..., 6 is allowed; this leaves 5 choices. By the multiplication principle there are $\binom{5}{2} \cdot 6 \cdot 5$ favorable outcomes. Therefore the probability of throwing a pair and a triple is $\boxed{\frac{\binom{5}{2} \cdot 6 \cdot 5}{6^5}}$.

(b) **...to have exactly two pairs? (like 2, 2, 3, 3, 4 but NOT 4, 4, 3, 3, 3 or 5, 5, 5, 5, 6.)**

Again the total number of outcomes is 6^5. We need to count those outcomes that contain exactly two pairs. We can first choose the one number that *does not* occur in either pair; there are 6 choices for this. Then we determine which numbers occur in the pairs. For this we need to choose a collection of two numbers out of the 5 remaining ones; for this we have $\binom{5}{2}$ choices.

What is left is to count the number of rearrangements of these numbers. It doesn't matter which numbers we choose, so we can assume that the dice show 1,1,2,2,3 and we want to count how many ways there are to rearrange this sequence. There are 5 spots total which we need to fill with two 1's, two 2's and one 3; there are $\binom{5}{2,2,1}$ possibilities.

Finally using the multiplication principle there are $6 \cdot \binom{5}{2} \cdot \binom{5}{2,2,1}$ favorable outcomes, so the probability to throw exactly two pairs is $\boxed{\frac{6 \cdot \binom{5}{2} \cdot \binom{5}{2,2,1}}{6^5}}$.

(c) ...that there are at least four equal?

The total number of outcomes is 6^5. To have at least four equal dice we can either have exactly four equal or all five equal. If all five are equal, there are just 6 choices (1,1,1,1,1 through 6,6,6,6,6). If exactly four are equal, we can first choose the number that occurs four times, for which there are 6 possibilities. Then for the number that occurs just once there are $5 = 6 - 1$ remaining choices. Finally we need to say in which spot the number occurs which only appears once (so that the number which appears four times occurs in the other four spots); there are $5 = \binom{5}{1}$ choices.

Thus there are 6 outcomes with five equal dice and $6 \cdot 5 \cdot \binom{5}{1}$ with exactly four equal dice. This gives a total of $6 + 6 \cdot 5 \cdot \binom{5}{1}$ favorable outcomes, so the probability to have at least four equal dice is $\boxed{\dfrac{6+6\cdot 5\cdot\binom{5}{1}}{6^5}}$.

10. **Florian, Grigor and Rob have an old tradition of playing poker every Sunday afternoon. One day, desperate in his desire to buy a new Porsche, Rob decides to try to cheat and brings to the game a deck of cards identical to the one they are playing with. During one of the games he gets two aces. What is the probability that if he replaces two of his cards with aces from the remaining two suits (so that he has 4 aces) Florian and Grigor won't be able to tell (by which we mean that they won't have either of those aces in their own hands)?**

    We want to find the probability that neither Florian or Grigor has an ace. Let's first count the number of ways in which Rob can have two aces among his cards. For the two aces there are $\binom{4}{2}$ possibilities and there are $\binom{48}{3}$ possibilities for the remaining 3 cards which Rob has (all but the four aces). What is left is the number of ways Florian and Grigor can get five cards each from the remaining 47 and this number is $\binom{47}{5,5,37}$. So the total number of possible outcomes is $\binom{4}{2}\binom{48}{3}\binom{47}{5,5,37}$. If we impose the condition that Florian and Grigor have no aces the number of possibilities for their five cards becomes $\binom{45}{5,5,35}$. Thus the total number of outcomes is $\binom{4}{2}\binom{48}{3}\binom{47}{5,5,37}$ and the number of favorable ones is $\binom{4}{2}\binom{48}{3}\binom{45}{5,5,35}$. The probability we are looking for is the ratio which is $\binom{45}{5,5,35}/\binom{47}{5,5,37}$. If one uses the formula for the multinomial coefficients one can simplify the above to $\boxed{\dfrac{36\times 37}{46\times 47}}$. The last number is approximately 0.616.

11. (a) **What is the greatest common divisor of 65 and 104?**

    We use Euclid's algorithm to find the greatest common divisor of 65 and 104. First we divide 65 into 104. It goes just once and there is a remainder of 39:

$$104 = 1\times 65 + 39$$

    We can continue with 65 and 39, as they will have the same gcd as 104 and 65; and so on:

$$\begin{aligned} 65 &= 1\times 39 + 26 \\ 39 &= 1\times 26 + 13 \\ 26 &= 2\times 13 + 0. \end{aligned}$$

    This shows that $\boxed{13}$ is the greatest common divisor of 65 and 104.

    (b) **What is their least common multiple?**

    Using the answer to part (a), we know that

$$\operatorname{lcm}(65,104) = \frac{65\cdot 104}{\gcd(65,104)} = \boxed{\frac{65\cdot 104}{13}} \ (= 5\cdot 104).$$

    (c) **How many numbers between 100 and 1000 are divisible by neither 4 nor 10?**

    To figure out how many numbers there are between 100 and 1000 that are neither divisible by 4 nor 10, we use double subtraction. The answer will be the number of whole numbers between 100 and 1000 minus the number of those that are divisible by 4, minus those that are divisible by 10, *plus* those that are divisible by both 4 and 10. (We subtracted them twice, so we have to add them back in!)

    First, there are $1000-100+1=901$ numbers between 100 and 1000.

    Next, $100 = 4\times 25$ and $1000 = 4\times 250$ so that there are $250-25+1=226$ numbers between 100 and 1000 that are divisible by 4.

Similarly we count those that are divisible by 10: since $100 = 10 \times 10$ and $1000 = 10 \times 1000$ there are $100 - 10 + 1 = 91$ such numbers.

A number is divisible by both 4 and 10 precisely if it is divisible by $\text{lcm}(4, 10) = 20$. We can write $100 = 20 \times 5$, $1000 = 20 \times 50$ and we find that there are $50 - 5 + 1 = 46$ numbers between 100 and 1000 that are divisible by both 4 and 10.

The answer therefore is $\boxed{901 - 226 - 91 + 46}$ $(= 630)$.

12. **A school goes on a trip to Smily River with 6 school buses. Once they reach the river, the kids go on a boat trip in 21 boats . A teacher notices that both in the buses and in the boats there was an equal number of kids and she knows that there are no more than 60 kids in the school. How many kids are there on the trip?**

    The fact that there was an equal number of kids in the buses means that the number of kids is divisible by 6 and the fact that there was an equal number in the boats means that the number of kids is divisible by 21. The greatest common divisor of 6 and 21 is 3 and their least common multiple is $\frac{6 \times 21}{3} = 42$. From the above discussion the number of kids is divisible by 42 and we know that it is less than 60 so the total number of kids on the school trip is $\boxed{42}$.

13. (a) **Use the Euclidean algorithm to find the greatest common divisor of 15 and 63?**

    Divide 63 by 15 to obtain 4 with a remainder of 3. Hence $\gcd(15, 63) = \gcd(15, 3)$. Since 15 is a multiple of 3, we have $\gcd(15, 3) = 3$. Therefore $\gcd(15, 63) = \boxed{3}$.

    (b) **What is the least common multiple of 14 and 63?**

    We use the formula $\text{lcm}(A, B) = A \times B / \gcd(A, B)$. Here $A = 14$ and $B = 63$. We use the Euclidean algorithm to find $\gcd(14, 63)$. Divide 63 by 14 to obtain 4 with a remainder of 7. Hence $\gcd(14, 63) = \gcd(14, 7)$. Since 14 is a multiple of 7, we have $\gcd(14, 7) = 7$. Therefore $\gcd(14, 63) = 7$. We conclude that $\text{lcm}(14, 63) = \boxed{14 \times 63/7}$. We may simplify the fraction to $14 \times 9 = \boxed{126}$.

    (c) **Use the Euclidean algorithm to find the greatest common divisor of 14 and 65?**

    Divide 65 by 14 to get 4 with a remainder of 9;
    Divide 14 by 9 to get 1 with a remainder of 5;
    Divide 9 by 5 to get 1 with a remainder of 4;
    Divide 5 by 4 to get 1 with a remainder of 1.
    Since 4 is a multiple of 1, we conclude that $\gcd(14, 65) = \boxed{1}$.