Answers to the 2nd Midterm

1. (a) **Use the Euclidean algorithm to find the greatest common divisor of 51 and 60.**

$$\begin{aligned} 60 &= 1 \cdot 51 + 9 \\ 51 &= 5 \cdot 9 + 6 \\ 9 &= 6 + 3 \end{aligned}$$

and so the GCD is $\boxed{3}$.

(b) **Find whole numbers x and y such that**

$$51x + 60y = 12 \text{ and } x > 0.$$

$$\begin{aligned} 3 &= 9 - 6 \\ &= 9 - (51 - 5 \cdot 9) = 6 \cdot 9 - 51 \\ &= 6(60 - 51) - 51 = 6 \cdot 60 - 7 \cdot 51 \end{aligned}$$

so $12 = 24 \cdot 60 - 28 \cdot 51$.
We add

$$-51 \cdot 60 + 60 \cdot 51 = 0$$

to obtain

$$12 = -27 \cdot 60 + 32 \cdot 51$$

and so $x = \boxed{32}$, $y = \boxed{-27}$.

(c) **Now find two whole numbers x and y such that**

$$51x + 60y = 12 \text{ and } y > 0.$$

We can use our first answers: $x = \boxed{-28}$, $y = \boxed{24}$.

2. **This problem concerns the binomial coefficient**

$$\binom{25}{17} = \frac{25!}{17!8!}.$$

(a) **Does 23 divide $\binom{25}{17}$? Why or why not?**

We write

$$\binom{25}{17} = \frac{25.24.23.22.21.20.19.18}{8!}.$$

$\boxed{\text{Yes}}$, because 23 divides the numerator and 23 *does not divide the denominator*.

(b) **Find the highest power of 3 dividing $\binom{25}{17}$.**

To see that 3^3 does not divide the quotient of the formula given in part (a), it's not enough to note that $27 > 25$, because 3^4 divides the numerator of this formula. However, only 3^2 divides the denominator, and so $3^{4-2} = \boxed{3^2}$ is the highest power dividing the quotient.

3. (a) **Find the prime factorization of 780.**

$$780 = 2 \cdot 390 = 2 \cdot 10 \cdot 39 = 2^2 \cdot 5 \cdot 39 = 2^2 \cdot 3 \cdot 5 \cdot 13.$$

(b) **How many divisors of 780 are there?**
(In other words, how many positive whole numbers, including 1 and 780, divide 780?)

We have three choices (0, 1, 2) for the exponent on 2 and two choices (0, 1) for each of 3, 5, and 13, so there are

$$3 \times 2 \times 2 \times 2 = \boxed{24}$$

factors. Note that this already includes $1 = 2^0 \cdot 3^0 \cdot 5^0 \cdot 13^0$ and $780 = 2^2 \cdot 3^1 \cdot 5^1 \cdot 13^1$.

4. **Find the following.**

Note: your final answer in arithmetic mod n must be a number in the range $0, 1, 2, \ldots, n-1$.

(a) $(5+6) \times 2 \pmod 7$.

$$11 \times 2 \equiv 22 \equiv \boxed{1} \pmod 7.$$

(b) $5^{18} \pmod{17}$.

$$5^{18} \equiv 5^{16} \times 5^2 \equiv 1 \times 25 \equiv \boxed{8} \pmod{17}.$$

(c) $102^{50} \pmod{103}$.

$$(-1)^{50} \equiv ((-1)^2)^{25} \equiv 1^{25} \equiv \boxed{1} \pmod{103}.$$

(d) $\frac{4}{6} \pmod{11}$.

$6^{-1} \equiv 2$ (as $6 \times 2 \equiv 1$) and so $\frac{4}{6} \equiv \boxed{8} \pmod{11}$.

(e) $\frac{3}{10} \pmod{71}$.

$10^{-1} \equiv -7$ (as $10 \times 7 \equiv -1$) and so $\frac{3}{10} \equiv -21 \equiv \boxed{50} \pmod{71}$.

5. **George P. Farnwrinkle IV goes into J. Press and buys several \$47 shirts. He pays with \$20 bills, the smallest denomination any Farnwrinkle has ever handled. A look of alarm and befuddlement passes over his face as the clerk hands him \$7 in change, which he immediately hands to the first panhandler he sees in the square. He bought fewer than 25 shirts, as this is the maximum his manservant can carry at once. How many shirts did he buy?**

Note: You must show your reasoning to get full credit.

Suppose he handed b bills and obtained s shirts plus \$7 change. Then

$$20b = 47s + 7.$$

Working (mod 20) this says

$$0 \equiv 7s + 7 \pmod{20}$$

which is equivalent to

$$-7 \equiv 7s \pmod{20}$$

and so

$$s \equiv -1 \equiv 19 \pmod{20}.$$

The one representative of this equivalence class less than 25 is 19 itself, and so he bought $\boxed{19}$ shirts costing a total of $19 \times \$47 = \$893$.

Note that the congruence

$$47s \equiv 7 \pmod{20}$$

(leading to the incorrect solutions 1 or 21) corresponds to buying s many shirts for a multiple of \$20 plus \$7 more, not with \$7 change.