QR 28 - The Magic of Numbers
Midterm 1 Solutions

1. (a) **How many numbers are there between 200 and 398 inclusive?**

    $398 - 200 + 1 = \boxed{199}$.

   (b) **How many even numbers are there between 200 and 398 inclusive?**

    Even numbers between 200 and 398 are $200, 202, \ldots, 396, 398$; after division by 2 we obtain $100, 101, \ldots, 198, 199$. The answer is $199 - 100 + 1 = \boxed{100}$.

   (c) **How many odd numbers are there between 200 and 398 inclusive?**

    The number of odd numbers between 200 and 398 is the difference between the number of all numbers and the number of even numbers. The answer is $199 - 100 = \boxed{99}$.

    Alternate solution: Odd numbers between 200 and 398 are $201, 203, \ldots, 395, 397$. After subtracting 1 and dividing by 2 we obtain $100, 101, \ldots, 197, 198$. The answer is $198 - 100 + 1 = \boxed{99}$.

   (d) **How many multiples of 4 are there between 200 and 398 inclusive?**

    The multiples of 4 between 200 and 398 are $200, 204, \ldots, 392, 396$. After division by 4 we obtain $50, 51, \ldots, 98, 99$. The answer is $99 - 50 + 1 = \boxed{50}$.

   (e) **How many numbers between 200 and 398 inclusive are even but not multiples of 4?**

    The number of even numbers that are not divisible by 4 is the difference between the number of even numbers and the number of the multiples of 4. The answer is $100 - 50 = \boxed{50}$.

    Alternate solution: The numbers in question are $202, 206, \ldots, 394, 398$. After subtracting 2 and dividing by 4 we get $50, 51, \ldots, 98, 99$. The answer is $99 - 50 + 1 = \boxed{50}$.

2. **A small town needs only five digit-long phone numbers, and additionally requires that numbers do not start with a zero. So, for example, 10983 or 97310 are allowed, but 08121 or 00123 are not. How many possible phone numbers:**

   (a) **contain exactly one digit 0?**

    If it contains *exactly* one digit 0, then since the initial cannot be 0, there are 4 possible places to put the unique 0. After 0 is assigned a position, for each of the 4 leftover places we can put one digit from $\{1,2, \ldots, 9\}$. By the multiplication principle, the number of final outcomes is $\boxed{4 \cdot 9^4}$.

   (b) **contain at least one digit 0?**

    The number of total five digit-long numbers with initial nonzero is $9 \cdot 10^4$. If further it does not contain 0, there are 9^5 such numbers. By the subtraction principle, there are $\boxed{9 \cdot 10^4 - 9^5}$ possible numbers that contain at least one digit 0.

   (c) **contain at least one digit 4?**

    The idea is the same as part (b). The number of total five digit-long numbers with initial nonzero is $9 \cdot 10^4$. If it does not contain 4, there are $8 \cdot 9^4$ such numbers, because the first digit can be only put a number from $\{1, 2, 3, 5, 6, 7, 8, 9\}$ without 0 and 4. By the subtraction principle, there are $\boxed{9 \cdot 10^4 - 8 \cdot 9^4}$ possible numbers that contain at least one digit 4.

3. **George and Barbara are getting lunch at a food court. George is going to 'Chicken King' while Barbara is going to 'Salad Express'.**

(a) **'Chicken King' offers a combination meal consisting of a chicken burger, fries and a soda. There are 4 different kinds of chicken burger, 2 different kinds of fries, and 8 different sodas. How many possible combination meals does George have to choose from?**

By the multiplication principle, there are $\boxed{4 \times 2 \times 8 = 64}$ combination meals for George to choose from.

(b) **Barbara decides to get the Basic Salad at 'Salad Express'. For the Basic Salad, you have to pick 5 different vegetables and 1 salad dressing. There are 11 vegetables and 5 salad dressings to choose from. How many different Basic Salads does Barbara have to choose from?**

The choice of vegetables and salad dressing is independent, so we can count Barbara's choices by the multiplication principle. She has to choose 5 vegetables from a total of 11 choices. The order that she chooses them in doesn't matter: she'll end up with the same salad regardless of the order. So there are $\binom{11}{5}$ ways to choose vegetables. Similarly, there are $\binom{5}{1}$ ways to choose the salad dressing. So the total number of salad choices is

$$\boxed{\binom{11}{5} \cdot \binom{5}{1}}.$$

The most common mistake with this problem was to use the formula for an ordered choice in choosing the vegetables, (which would yield the incorrect answer $11 \times 10 \times 9 \times 8 \times 7 \times 5$).

(c) **After his 'Chicken King' meal, George decides to get an ice cream sundae from 'Sundae Shop'. He gets to choose 3 scoops of ice cream from a choice of 7 flavors, but the scoops do not all have to be different flavors (for example, he could pick 3 scoops of vanilla if he wants). How many ice cream sundaes does George have to choose from?**

George is choosing 3 flavors from 7, with the possibility of repetition. However, as is implied in the problem (and Dick explicitly stated during the exam), the order of the three scoops does not affect the sundae that results. As we've learned in class, the formula for choosing a collection of k objects with repetition from a group of n choices is $\binom{n+k-1}{k}$. In our case, this formula yields $\boxed{\binom{9}{3} = \binom{9}{6}}$. We can picture this as choosing 6 empty trays to separate the 3 flavors (hence from a total of $6 + 3$).

Some people did the problem as if the order of the scoops chosen mattered (which would yield the incorrect answer 7^3). Some people wrote $\binom{7}{3}$, which would be the correct answer if one was not allowed to repeat flavors. Many got the formula wrong: $\binom{10}{3}$, $\binom{9}{2}$ and $\binom{9}{7}$ were all popular answers.

4. (a) **A group of 11 Harvard math majors get together to form a soccer team. To do this they divide themselves into 4 groups: 4 defenders, 4 midfielders, 2 strikers and a goalkeeper. How many ways are there to do this?**

This is a straightforward example of multinomials; the answer—the number of ways of dividing a poll of 11 into collections of size 4, 4, 2 and 1—is just

$$\boxed{\binom{11}{4, 4, 2, 1} = \frac{11!}{4!4!2!1!}}.$$

Alternatively, you can choose the groups one at a time: for example, you can choose the 4 defenders first, then choose 4 midfielders from among the 7 remaining math majors, and finally choose the 2 strikers from among the 3 left; this yields the answer

$$\boxed{\binom{11}{4}\binom{7}{4}\binom{3}{2}}$$

which is the same number.

(b) **The same 11 math majors want to form an indoor soccer team with only 5 players. They need to pick 5 players to be in the team, and they need to pick one of those to be the goalkeeper (the other positions are interchangeable for indoor soccer). How many ways are there to form such a team?**

Again, there are several ways to set this up, all of which yield the same answer in different forms. For one, you can say you're dividing the 11 math majors into three groups: 1 goalie, 4 players other than the goalie, and 6 nonplayers; this gives

$$\boxed{\binom{11}{1,4,6} = \frac{11!}{4!6!}}.$$

Alternatively, you can choose which 5 of the 11 majors will be on the team, then choose which one of those five will be the goalie; this gives

$$\boxed{\binom{11}{5} \cdot 5};$$

or you could choose the goalie first, then choose the 4 nongoalie players from the 10 math majors remaining, to arrive at

$$\boxed{11 \cdot \binom{10}{4}}.$$

You should check that all these answers are in fact the same.

5. (a) **You're dealt four cards from a standard deck of 52 cards (13 different cards in each of 4 suits). What's the probability of being dealt a hand where all four cards are of the same suit?**

Let's consider hands where we don't care about order. There are $\binom{52}{4}$ such 4-card hands. Now let's consider which of these have all the same suit. There are 4 suits, and $\binom{13}{4}$ ways of choosing 4 cards within a given suit. Thus the probability is

$$\boxed{\frac{4 \cdot \binom{13}{4}}{\binom{52}{4}}}.$$

Another way to solve this is to consider the deal in order. There are $52 \cdot 51 \cdot 50 \cdot 49$ ways to deal 4 cards. Any of 52 cards could be the first card of a same-suit deal, but then only 12 cards of those that remain are the same suit, and then 11 for the third card, and finally 10 possibilities for the last card to remain in the same suit as the first three. Thus the probability is

$$\boxed{\frac{52 \cdot 12 \cdot 11 \cdot 10}{52 \cdot 51 \cdot 50 \cdot 49}}.$$

You should check that these two answers are the same.

(b) **In another game, you and a friend are each dealt a hand of two cards. You are using a standard deck of 52 cards. The best hand is a pair of aces. What's the probability that you'll both be dealt a pair of aces?**

Consider the four cards lying on the table. Our problem is equivalent to asking the probability of that pile being precisely the four Aces. There is one 4-card hand of all Aces out of $\binom{52}{4}$, giving a probability of $\boxed{\frac{1}{\binom{52}{4}}}$.

Suppose we instead consider first the friend's cards and then your cards. There are $\binom{52}{2}$ 2-card deals to him, following which there are $\binom{50}{2}$ 2-card deals from the remainder to

you. Of these, there are $\binom{4}{2}$ ways your friend could be dealt two of four Aces, and $\binom{2}{2} = 1$ way for you to get the remaining 2 Aces. In total, the probability is

$$\boxed{\frac{\binom{4}{2} \cdot \binom{2}{2}}{\binom{52}{2} \cdot \binom{50}{2}}}.$$

Finally, we could look at the deal in order as with the previous problem. There are again $52 \cdot 51 \cdot 50 \cdot 49$ ways to deal 4 cards. The first card could be any of 4 Aces, the second any of the 3 remaining Aces, the third any of 2, and the fourth must be the 1 remaining Ace. Thus the probability is $\boxed{\frac{4 \cdot 3 \cdot 2 \cdot 1}{52 \cdot 51 \cdot 50 \cdot 49}}$. You should again check that these three answers are the same.