

Practice Midterm 1 Solutions

1. (a) **How many numbers are there between 1776 and 2005 (inclusive) which are divisible by 2 but not by 4?**

For divisibility by 2, first lop, so we want the totality of numbers divisible by 2 between 1776 and 2004 which is the same *totality* as the numbers between 888 and 1002; i.e.,

$$1002 - 888 + 1 = 115$$

but we must throw out the numbers between 1776 and 2004 that are divisible by 4 (i.e., the even numbers between 888 and 1002 and there are

$$501 - 444 + 1 = 58$$

of them (why?) which gives a total of

$$115 - 58 = \boxed{57}$$

as a final answer. There is also a slightly different way of proceeding, by thinking of the numbers divisible by 2 and not by 4 as numbers of the form $4n + 2$ and organizing the count that way: when you lop you can start with 1778 which is the first number divisible by 2 and not by 4 in the bunch and end with 2002; now every fourth number in the sequence between 1778 and 2002 is divisible by 2 and not by 4, and this accounts for all of them. So, we count $1 + \frac{2002-1778}{4} = 1 + \frac{224}{4} = 1 + 56 = \boxed{57}$.

- (b) **How many numbers in the previous problem are not divisible by 3?**

Here, noting that by the first part of this question there are 57 numbers divisible by 2 but not by 4 in the range between 1778 and 2002 (inclusive) so let us consider the ones in this range that are divisible by 3, and toss them out to get our final answer. 1782 is the first in this pile divisible by 3, 1998 is the last; and from then on every third member of the 54 numbers remaining in our pile (between 1786 and 1998, inclusive) are similarly divisible by 3. This gives a count of $1 + \frac{54}{3} = 19$ numbers divisible by 3 that we must exclude. (Note: the 1 in $1 + \frac{54}{3}$ counts the beginning number 1786). So the final answer is

$$57 - 19 = \boxed{38}.$$

2. (a) **How many numbers are there between 60 and 150, inclusive?**

The famous formula gives it:

$$150 - 60 + 1 = \boxed{91}.$$

- (b) **Of the numbers in (a), how many are divisible by 2? How many are divisible by 3?**

The numbers divisible by 2 are $60 = 2 \times 30, 2 \times 31, \dots, 2 \times 75 = 150$; so there are $75 - 30 + 1 = \boxed{46}$ of them. Those divisible by 3 are $60 = 3 \times 20, 3 \times 21, \dots, 3 \times 50 = 150$; so there are $50 - 20 + 1 = \boxed{31}$ of them.

- (c) **Of the numbers in (a), how many are divisible by 2 but not by 3?**

We have to find the numbers divisible by 2 and take out all those that are also divisible by 3. A number is divisible by 2 and 3 precisely if it is divisible by 6; so let us find those. They are $60 = 6 \times 10, 6 \times 11, \dots, 6 \times 25 = 150$; so there are $25 - 10 + 1 = 16$ of them. Thus the answer is $46 - 16 = \boxed{30}$ (note that 46 was found in the previous part).

- (d) **Of the numbers in (a), how many are divisible by 2 or by 3 (or both)? (Examples of such numbers are 96, 99, 100.)**

Our first guess might be to add the number of those divisible by 2 to the number of those divisible by 3; this gives $46 + 31$ (by part (b)). But we have counted those numbers *twice* that are divisible by both 2 and 3 (i.e., the numbers divisible by 6), and we need to subtract the overcount. There are 16 numbers between 60 and 150 that are divisible by 6 (by part (c)). Thus the final answer is $46 + 31 - 16 = \boxed{61}$.

3. **The Greek alphabet has 24 letters, 7 of which (A, E, H, I, O, Υ, Ω) are vowels.**

- (a) **How many possible five-letter Greek words are there? (“Mathematical words”, that is: as usual we do not ask that the words make any sense — it’s all Greek to us in any case.)**

$\boxed{24^5}$ (Multiplication Principle: five slots, independent choice of 24 letters for each).

- (b) **Of the five-letter Greek words, how many contain exactly two vowels?**

To construct such a word, we first choose two slots out of five where the vowels will go, then choose two vowels to fill these slots, then choose three consonants to fill the remaining three slots. The first task can be done in $\binom{5}{2}$ ways. However these are chosen, the second task can be done in 7^2 ways, and the third in $(24 - 7)^3$ ways. So by the Multiplication Principle the total count is the product $\boxed{\binom{5}{2} \times 7^2 \times (24 - 7)^3}$.

- (c) **A *palindrome* is a word, such as SEES or DEIFIED or ΕΩΦΦΩΕ, that reads the same forward and back. Of the five-letter Greek words with exactly two vowels, how many are palindromes?**

Now the two vowels must go in slots 1 and 5, or 2 and 4, and be the same. So we have only 2×7 choices there. This leaves three consonants, but only two choices because the last consonant must be the same as the first. Therefore the total count is $\boxed{2 \times 7 \times (24 - 7)^2}$.

4. (a) **Suppose that a license plate has seven spots, and that valid characters for these slots are the 26 letters (A through Z) and the 10 digits (0 through 9). How many possible license plates are there?**

There are 36 choices (26 letters plus 10 digits) for each of seven spots. Thus the multiplication principle tells us that the total number of choices is $\boxed{36^7}$.

- (b) **Now suppose that you want your license plate to contain exactly 3 letters. How many possible plates are there?**

First, we need to choose which spots get letters and which get digits; we can think of this as determining the “template” for the solution. We can do this by choosing which 3 spots get letters (the rest will get digits); there are $\binom{7}{3}$ ways of doing this. After we’ve chosen where the letters go and where the digits go, we need to choose which letters and digits will be used. There are 26 choices for each of 3 letters, so there are 26^3 ways to choose the letters. Similarly, there are 10^4 ways to choose the 4 digits. Hence there are $26^3 \cdot 10^4$ ways to choose the particular letters and digits used. Since each of our $\binom{7}{3}$ templates gives $26^3 \cdot 10^4$ license plates, the total number of possible plates is $\boxed{\binom{7}{3} 26^3 \cdot 10^4}$.

- (c) **Now suppose that, in addition to having exactly 3 letters, you also refuse to allow these letters to be all in a row. How many possible plates are there meeting these conditions?**

We will use the subtraction principle. We wish to take the number of plates with exactly 3 letters (which we know from the last part) and then subtract the number of such plates with all 3 letters in a row. Since we’re now interested in the number of plates with all 3 letters together, we can think of these 3 letters as forming a single block. Thus we have 5 slots which we wish to fill with 4 digits and a block of letters. There are $\binom{5}{1} = 5$ ways

of doing so, since we simply need to pick a slot for the letters. So there are 5 templates when we want all 3 letters together. Once we have our template, we need to pick specific digits and letters. Just as before, there are $26^3 \cdot 10^4$ ways to do this (since we still have 3 letters and 4 digits). We conclude that there are $5 \cdot 26^3 \cdot 10^4$ plates with all 3 letters together. Using this and the solution to the previous part in our subtraction principle, we see that the number of plates (with exactly 3 letters) such that all 3 letters aren't

together is
$$\boxed{\binom{7}{3} 26^3 \cdot 10^4 - 5 \cdot 26^3 \cdot 10^4 = \left(\binom{7}{3} - 5 \right) 26^3 \cdot 10^4}.$$

5. (a) **You are dealt, at random, 4 cards from a standard 52 card deck. What is the probability that the denominations are all consecutive (that is, you have the 4-card version of a straight)? Here the Ace can be either high or low, so that $\{A, 2, 3, 4\}$ and $\{J, Q, K, A\}$ both count.**

We can think of our outcomes as a collection of 4 cards out of the 52; these are equally likely. Then there are $\binom{52}{4}$ total outcomes. Next we need to count the favorable outcomes. We first choose the denominations in our "4-card straight." The lowest denomination can be anything from an Ace (low) up to a Jack (any higher and 4 cards won't fit). Thus there are 11 choices for the denomination of the low card. Once we choose the low card, the rest of the denominations are determined as well, since they must be consecutive. Once we've chosen our denominations, we must give each card a suit. Since there are 4 cards and each can be any of 4 denominations, there are 4^4 ways to assign suits to our cards. So by first choosing the denominations and then the suits, we see that there are $11 \cdot 4^4$ possible "4-card straights." Since these are our favorable outcomes, we divide by the total outcomes to find that the probability of being dealt a "4-card straight" is

$$\boxed{\frac{11 \cdot 4^4}{\binom{52}{4}}}.$$

- (b) **What is the probability that you have 2 pair, that is, that you have two cards of one denomination and two cards of a different denomination?**

Again, the total number of outcomes is $\binom{52}{4}$. We proceed to count the number of favorable outcomes, that is, the number of 2 pair hands. We start by choosing denominations for our cards. Out of the 13 denominations, we need 2 of them to be the denominations of our 2 pairs. Thus, there are $\binom{13}{2}$ ways of picking the denominations. Next we need to choose suits for our cards. For the first pair, there are 4 possible suits, of which we must choose 2. So there are $\binom{4}{2}$ ways to assign suits to the first pair. Similarly, there are $\binom{4}{2}$ ways to assign suits to the second pair, and thus there are $\binom{4}{2}^2$ ways to assign suits to all 4 cards. By first assigning denominations to our cards and then suits, we see that there are $\binom{13}{2} \binom{4}{2}^2$ possible 2 pair hands, and thus that many favorable outcomes as well. Dividing by the total number of outcomes, we see that the probability of being dealt 2

pair is
$$\boxed{\frac{\binom{13}{2} \binom{4}{2}^2}{\binom{52}{4}}}.$$

6. **You are playing a slightly unorthodox version of Scrabble where you are given ten letters (rather than the usual seven). Amazingly enough, you drew (randomly) from the pile precisely these ten letters: ACCELERATE, but probably not in exactly that order. Given that you drew this set of ten Scrabble tiles, what is the probability that you did, in fact, draw them from the pile spelling the word ACCELERATE?**

To distinguish between the tiles showing the same letter, let's label them $A_1, A_2, C_1, C_2, E_1, E_2, E_3, L, R, T$. An outcome is an ordering of the ten tiles which you draw. So there are 10! outcomes.

How many orderings spell out the word ACCELERATE? Well, the ordering is determined by

this word, except that you have $2!$ ways to arrange the A's, $2!$ ways to arrange the C's and $3!$ ways to arrange the E's. This gives a total of $3! \cdot 2! \cdot 2! = 24$ choices, by the multiplication principle.

(Alternatively: there are two choices for the first A (A_1 or A_2), two choices for the first C (C_1 or C_2), one remaining choice for the second C, three choices for the first E, one choice for the L, two remaining choices for the second E, ... By the multiplication principle there are $2 \cdot 2 \cdot 1 \cdot 3 \cdot 1 \cdot 2 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 24$ ways of ordering the ten tiles to spell out ACCELERATE.)

Thus the answer is

$$\frac{3! \cdot 2! \cdot 2!}{10!}.$$

Note: Alternatively you could take as an outcome the actual word formed by these tiles in the order drawn. There is one favorable outcome out of a total of $\binom{10}{3,2,2,1,1,1}$, which gives the same answer. It is not as intuitive, however, that each word formed by the Scrabble tiles should be equally likely.

7. **Yahtzee is played with 5 dice, each numbered in the usual way from 1 to 6 and equally likely to land on each face.**

- (a) **What is the probability of the total on the 5 dice being at least 29?**

Note that the maximum score is $5 \times 6 = 30$, so "at least 29" can be achieved with all 6's, or four 6's and one 5, and in no other way.

There are 6^5 equally likely throws of five honest dice. Of those, the favorable ones are the single all-6 throw and the $\binom{5}{1} = 5$ throws where one die shows 5 and the others all show 6. So the probability is $\frac{1+5}{6^5}$ (which you may simplify to $1/6^4$).

- (b) **Suppose now that one of the dice is marked dishonestly, with the 1-face replaced by a second 6 (so the faces are 2,3,4,5,6,6 instead of the honest 1,2,3,4,5,6). What is the probability of scoring at least 29 with this die and four honest dice?**

There are still 6^5 equally likely throws of these dice. You have several ways to organize the computation of the number of favorable throws; here is one. The dishonest die must show 6 (two faces) or 5 (one face) for the total to be at least 29. In the first case, the others must either be all 6 (one possibility), or one 5 and three 6's ($\binom{4}{1} = 4$ possibilities), for a total of $2 \times (1 + 4) = 10$. In the second case, there is only one favorable outcome for the remaining four dice: all must show 6's. Adding this one favorable outcome to our previous 10 yields a total of 11, so the probability is now $\frac{11}{6^5}$.

8. **At parties, local mathemagician Polly Nomial often produces an ordinary pack of cards and invites a stranger to choose three at random. Polly will then claim that at least one of the chosen cards is an ace. The thing is: there's no trick involved! What's the probability that Polly's claim is correct (i.e., that at least one of the three is an Ace)?**

The total number of three-card hands is $\binom{52}{3}$ and the total number of three-card hands *without* aces is $\binom{48}{3}$. Hence the total number of hands with *at least one* ace is $\binom{52}{3} - \binom{48}{3}$.

So we see that Polly is correct with probability

$$\frac{\binom{52}{3} - \binom{48}{3}}{\binom{52}{3}}.$$

9. (a) **Compute $\binom{16}{3}$ as a whole number.**

$$\binom{16}{3} = \frac{16 \cdot 15 \cdot 14}{3 \cdot 2} = 8 \cdot 5 \cdot 14 = \boxed{560}.$$

(Here we just cancelled opportunistically, before doing the multiplication.)

- (b) **Compute $\binom{17}{14}$ as a whole number.**

Since $\binom{17}{14} = \binom{17}{3}$ we proceed as above:

$$\binom{17}{3} = \frac{17 \cdot 16 \cdot 15}{3 \cdot 2} = 17 \cdot 8 \cdot 5 = \boxed{680}.$$

- (c) **Yes or no, and why: Is $\binom{106}{102}$ bigger than $\binom{105}{4} + \binom{105}{3}$?**

They are the same: $\binom{106}{102} = \binom{106}{4}$ and for either of the two reasons discussed in class $\binom{106}{4} = \binom{105}{4} + \binom{105}{3}$.

10. **You go shopping at the Pen Store, and need to buy 10 pens. There are blue pens, red pens, and orange pens. In how many ways can you choose 10 of them?**

Consider 10 pens with two empty trays in between. The pens to the left of both trays will be blue, those in between the trays will be red, and those to the right of both will be orange.

There are $\boxed{\binom{10+2}{2}}$ ways to choose the location of the two trays among the $10+2$ items.