

# Homework 15 Solutions

## Problems

1. Note that in this question you do not need to find the full prime factorization of the number!

- (a) What is the largest prime number that divides  $\binom{26}{7}$ ?  
(b) What is the largest prime number that divides  $\binom{26}{7}$  twice (i.e., whose square divides it)?

$$\binom{26}{7} = \frac{26 \times 25 \times 24 \times 23 \times 22 \times 21 \times 20}{7 \times 6 \times 5 \times 4 \times 3 \times 2}$$

We see that the largest prime dividing the top line is 23 and this doesn't divide the bottom. Hence the largest prime dividing  $\binom{26}{7}$  is  $\boxed{23}$ .

The largest prime dividing the top line at least twice is 5. 5 divides the top line 3 times and the bottom line once. Hence  $5^2$  divides  $\binom{26}{7}$  so the largest prime dividing  $\binom{26}{7}$  twice is  $\boxed{5}$ .

2. Find the prime factorization of

$$(3^{13} \times 5^{14} \times 11) - (3^{11} \times 5^{12} \times 11^3).$$

$$\begin{aligned} & (3^{13} \times 5^{14} \times 11) - (3^{11} \times 5^{12} \times 11^3) \\ &= (3^{11} \times 5^{12} \times 11)(3^2 \times 5^2 - 11^2) \end{aligned}$$

Now,  $3^2 \times 5^2 - 11^2 = 225 - 121 = 104 = 4 \times 26 = 8 \times 13$ , so the prime factorization is  $(3^{11} \times 5^{12} \times 11) \times (8 \times 13) = \boxed{2^3 \times 3^{11} \times 5^{12} \times 11 \times 13}$ .

3. Let  $a$  be such that  $\binom{21}{10} = 2^a \times 3 \times 7 \times 13 \times 17 \times 19$ . Find  $a$ .

We have

$$\binom{21}{10} = \frac{21 \times 20 \times 19 \times 18 \times 17 \times 16 \times 15 \times 14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 65 \times 4 \times 3 \times 2 \times 1}{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$$

Writing out just the even terms gives us

$$\frac{\dots \times 20 \times \dots \times 18 \times \dots \times 16 \times \dots \times 14 \times \dots \times 12 \times \dots \times 10 \times \dots \times 8 \times \dots \times 6 \dots \times 4 \times \dots \times 2 \times \dots}{\dots \times 10 \times \dots \times 8 \times \dots \times 6 \times \dots \times 4 \times \dots \times 2 \times \dots \times 10 \times \dots \times 8 \times \dots \times 6 \times \dots \times 4 \times \dots \times 2 \times \dots}$$

Now we consider the powers of two contained in these terms:

$$\frac{4 \times 2 \times 16 \times 2 \times 4 \times 2 \times 8 \times 2 \times 4 \times 2 \times \dots}{2 \times 8 \times 2 \times 4 \times 2 \times 2 \times 8 \times 2 \times 4 \times 2} = \frac{2^{2+1+4+1+2+1+3+1+2+1}}{2^{1+3+1+2+1+1+3+1+2+1}} = \frac{2^{18}}{2^{16}} = 2^{\boxed{2}}$$