

MATH E-105, SPRING 2001  
GROUPS, GRAPHS, AND ALGEBRAIC STRUCTURES FOR  
COMPUTING  
Homework Assignment # 2  
Due: July 2, 2001

Reading

This is essentially the same reading assignment as Monday's, with some additions in sections 4.1, 9.4, and 9.5. (Try to understand the theorems with examples before attempting the proofs.)

- (background) Section 2.7 (pp. 88–91), especially 2.51–2.55.
- Section 3.6 (pp. 129–132), especially 3.51–3.53, 3.55, 3.57, and 3.61–3.62.
- Section 4.1 (pp. 136–139), especially Definitions 4.7 and 4.13.
- Section 4.2 (pp. 141–146), especially Examples 4.16 and 4.18.
- Section 9.4 (pp. 380–385).
- Section 9.5 (pp. 389–390).

Required Problems

1. Do exercises #8,9,12 from pp. 134–135, section 3.6.
2. Do exercises #1,2,9 from pp. 385–386, section 9.4.
3. Make a table listing the multiplicative inverse for every non-zero element of  $\mathbb{Z}_{13}$ .
4. Do exercise #10 from p. 392, section 9.5.
5. The symmetry group of the equilateral triangle ( $D_3$ ), the permutation group on the set of three letters ( $S_3$ ), and the group of additive congruences modulo 6 ( $\mathbb{Z}_6$ ) all have six elements. Are these different manifestations of the same group? Which of them are abelian? Which are cyclic? Find a natural identification between the two that are isomorphic.

6. Let  $\sigma = (1357)(246)$  be a permutation of the set of seven elements.
- Find  $\sigma^2$  and  $\sigma^3$ .
  - What is the order of  $\sigma$ ?
  - Find  $\sigma^{-1}$ .
  - How many permutations of the set of seven elements are there?
  - What is the largest order of an element of  $S_7$ ? Explain.
7. One labeling of the regular hexagon is the figure below, where any symmetry operation on the hexagon permutes the sets of elements that share a number. For example, the three edges numbered 4 may be interchanged with the three edges numbered 5.
- Write the five permutations from  $S_5$  that represent non-trivial rotations about an axis perpendicular to the plane of the hexagon.
  - Write the three permutations from  $S_5$  that represent reflections about axes between two opposite vertices (or, if you prefer,  $180^\circ$  rotations about these axes).
  - Express the last three non-trivial symmetries of the regular hexagon as permutations from  $S_5$ , and describe what each does geometrically.
8. The abelian group  $V_4$  consists of four elements  $i, a, b,$  and  $c$ , where  $i$  is the identity, each of the others has order 2, and the product of any two non-identity elements is the third non-identity element.
- Find both of the subgroups of  $D_4$ , the symmetry group of the square, that are isomorphic to  $V_4$ .
  - Find one subgroup of the symmetry of the regular tetrahedron that is isomorphic to  $V_4$ .

### Exploratory Problems

9. Show that  $|D_n| = 2n$ , that  $|S_n| = n!$ , and that  $|\mathbb{Z}_n| = n$ .
10. Recall the definition of the four-element group  $V_4$  from Problem # 8. Identify subgroups of the symmetry groups of the cube and icosahedron that are isomorphic to  $V_4$ .
11. Describe the symmetry group of the circle.
12. Classify the capital letters of the English alphabet by their symmetry groups. (What assumptions are you making about your font?)
13. Prove that if a positive integer  $n$  is not prime, then  $(\mathbb{Z}_n^\times, \cdot)$  is not a group.
14. Show that  $D_5$ , the symmetry group of the pentagon, is isomorphic to a subgroup of  $S_5$  by explicitly listing the ten elements of  $S_5$ .
15. Consider the following six functions:

$$\begin{array}{ll} f_1(x) = x & f_2(x) = 1 - x \\ f_3(x) = \frac{1}{x} & f_4(x) = \frac{1}{1-x} \\ f_5(x) = \frac{x}{x-1} & f_6(x) = \frac{x-1}{x} \end{array}$$

Now think of these as a group under the operation composition. For example,

$$\begin{aligned} (f_2 \circ f_3)(x) &= f_2(f_3(x)) \\ &= f_2\left(\frac{1}{x}\right) \\ &= 1 - \frac{1}{x} \\ &= \frac{x-1}{x} \\ &= f_6(x). \end{aligned}$$

What group is this?