MATH E-105, SPRING 2001
GROUPS, GRAPHS, AND ALGEBRAIC STRUCTURES FOR COMPUTING
Homework Assignment # 4
Due: July 11, 2001

Reading

- Sections 9.4 and 9.5, plus hand-outs.

Required Problems

1. Problem #1 from p. 391 of Section 9.5.
2. Problem #8 from p. 392 of Section 9.5.
3. Show that $H = \{I, (12)\}$ is not a normal subgroup of S_3 by computing its conjugate subgroups, and interpret this result geometrically.
4. Show that $J = \{I, (123), (132)\}$ is a normal subgroup of S_3 by computing its conjugate subgroups, and interpret this result geometrically.
5. Consider the map $\varphi : S_5 \longrightarrow S_4$ that deletes the character 5 from consideration. For example, $\varphi((1354)) = (134)$ and $\varphi((123)) = (123)$. Is this map a homomorphism? Why or why not? If so, what is its kernel?
6. Consider the map $\psi : S_5 \longrightarrow \mathbb{Z}_2$ (where the group operation for $\mathbb{Z}_2$ is addition) that takes even permutations to the element 0 and odd permutations to the element 1. Is this map a homomorphism? Why or why not? If so, what is its kernel?
7. Exhibit an isomorphism between the non-zero elements of $\mathbb{F}_8$ (that is, the multiplicative group of the field) and the elements of $\mathbb{Z}_7$ (considered as an additive group).
8. In this exercise, you will complete the identification of C_3 as the quotient group of A_4 by its normal subgroup V_4.

   Exhibit V_4 as a (normal) subgroup of the rotation group of the tetrahedron. List as permutations the members of the three cosets. Identify each coset with an element of C_3. For each pair of cosets (recall that C_3 is abelian!), perform the multiplication by members of the cosets chosen at random, and thereby reconstruct the multiplication table for C_3.

Exploratory Problems

9. Show that if $\varphi : G \longrightarrow H$ is a homomorphism, then $\ker(\varphi)$ is a normal subgroup of G.

10. Exhibit an isomorphism between the real numbers under addition and the positive real numbers under multiplication.

11. How many generators does the group C_n have? *(Hint: You might first consider the case when n is prime.)*

12. Find sets of generators for the groups D_n (2 generators) and S_n ($n-1$ generators). (The point here is to this *generally.* Come up with an abstract way of writing these down in a way that treats all cases simultaneously.)

13. Let S_{cube} be the symmetry group of the cube. Prove that $S_4 \cong S_{\text{cube}}$ by doing the following:

    - Write down the elements of S_{cube}. (How many symmetries are there? How many permutations are there in S_4?)
    - Construct a (hollow) model of the cube, and label the four axes between pairs of opposite vertices.
    - Show that there is a symmetry of the cube that interchanges two of these axes while leaving the other two fixed. *Note that "fixing an axis" does not imply that the axis remains unmoved, but simply that it winds up where it started. For example, an axis can end up facing the other direction but otherwise in its original position.*
    - Write down an isomorphism $\varphi : S_4 \longrightarrow S_{\text{cube}}$ that takes transpositions to the symmetries described above, and argue that you can thus generate all symmetries of the cube.
    - Verify that the group structure is preserved by your isomorphism, i. e. that $\varphi(ab) = \varphi(a)\varphi(b)$ for every a and b.