MATH S-15, SUMMER 2001
GROUPS, GRAPHS, AND ALGEBRAIC STRUCTURES FOR COMPUTING
Homework Assignment # 5
Due: July 16, 2001

Reading

- Section 4.3 (pp. 147–155), especially 4.31–4.42 and 4.52–4.54.
- Section 5.8 (pp. 217–222).
- Section 8.11 (pp. 359–362).

Required Problems

1. Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = xy$.

   Is f a linear function? Explain.

2. Do exercises # 1cde and # 4ef from p. 155, Section 4.3.
3. Do exercise # 6 from p. 156, Section 4.3.
4. Do exercise # 4bc from p. 223, Section 5.8.
5. Find a group of 2×2 matrices (under multiplication) that is isomorphic to D_4.
6. Find the matrix representation for a reflection in $\mathbb{R}^2$ through the lines $y = -x$ and $y = 2x$.
7. Find the matrix representation for a reflection in $\mathbb{R}^3$ through the plane $y = x$.
8. Let $A = \left[\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right]$. Find a vector $\mathbf{v} \in \mathbb{R}^{\mathbf{2}}$ such that $A\mathbf{v} = \lambda\mathbf{v}$ for some scalar $\lambda \in \mathbb{R}$. What is the value of λ? Are there any other vectors that have this property?
9. Show that $\det(A) = \det(A^t)$ if A is a 2×2 matrix. Show that this is also true if A is a 3×3 matrix.

10. Match each of the eight 2×2 matrices in the left-hand column with its geometrical behavior listed in the right-hand column.

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$H = \begin{bmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{bmatrix}$$

- rotation by 180°
- clockwise rotation by 90°
- counter-clockwise rotation by 90°
- reflection about the line $y = x$
- reflection about the y-axis
- shear parallel to the x-axis
- collapses everything onto the x-axis
- stretches along the line $y = x$ and shrinks along the line $y = -x$

Exploratory Problems

11. Find the matrix representations for rotations in $\mathbb{R}^3$ by the angles 60°, 90°, and 180° about the line defined by the equations $y = x$ and $z = 0$.

12. Find the multiplicative inverse for the matrix $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. *(Hint: Not all* 2×2 *matrices have multiplicative inverses. What are the conditions for* A *to have an inverse?)*

13. Use trigonometry to prove the rule:

$$\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||\ ||\mathbf{v}||\ \cos\theta,$$

where θ is the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$.

14. Find the coordinates in $\mathbb{R}^3$ for a reasonable orientation of the regular octahedron. *(Hint: There is a very nice way to do this! Think about the axes through the vertices.)* Write down a 3×3 matrix that performs:

    - a rotation by 90° about a vertex-vertex axis
    - a rotation by 180° about an edge-edge axis
    - a rotation by 120° about a face-face axis

    Now, using multiplication of matrices and facts about the symmetry group of the octahedron, S_4, write down the 24 elements of the symmetry group in matrix form.

15. Let $M_2(\mathbb{R})$ be the set of 2×2 matrices with real entries. Show that the determinant function

$$\det : M_2(\mathbb{R}) \longrightarrow \mathbb{R}$$

satisfies the basic homomorphism rule: $\det(AB) = \det(A) \cdot \det(B)$.

(For both $M_2(\mathbb{R})$ and $\mathbb{R}$, we consider the operation multiplication. Note that *neither* of these is a group under multiplication, but that the rule still holds.)

16. Construct a vector space defined over the field $\mathbb{Z}_p$ by modelling what we did in constructing the vector space $\mathbb{R}^2$ over the field $\mathbb{R}$. What are the vectors in this vector space? What are the "lines" in this vector space? Is there a norm or a dot product in this vector space? How would you define angles?