MATH S-15, SUMMER 2001
GROUPS, GRAPHS, AND ALGEBRAIC STRUCTURES FOR COMPUTING
Homework Assignment # 6
Due: July 18, 2001

Reading

Sections 4.3, 5.8, and 8.11, and hand-outs.

Required Problems

1. Do exercise # 3 from p. 223, Section 5.8.
2. Do exercise # 6ab from p. 223, Section 5.8.
3. Do exercise # 3 from p. 363, Section 8.11, but also discuss the long-term behavior of the system.
4. Show that if $\mathbf{u}$ and $\mathbf{v}$ are both eigenvectors for the matrix $A$ with the same eigenvalue $\lambda$, then $\mathbf{u}+\mathbf{v}$ is also an eigenvector for $A$ with eigenvalue $\lambda$.
5. Consider the matrix $A = \left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right]$. Find all eigenvalues and eigenvectors of $A$ when:
   - $a = d$ and $b = 0$
   - $a = d$ and $b \neq 0$
   - $a \neq d$ and $b = 0$
6. Find all eigenvalues and eigenvectors of the matrices
$$A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$
7. Let $A = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ be the reflection about the line $y = x$, and let $B = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ be rotation by $90^\circ$. Find the eigenvalues and eigenvectors of $A$ and $B$, and interpret these results geometrically.
8. Find explicit conditions on the entries of the matrix $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ such that $A \in O(2, \mathbb{R})$. *(Hint: Consider the equation $||A\mathbf{v}|| = ||\mathbf{v}||$ for the vector $\mathbf{v} = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$.)*

Exploratory Problems

9. Do exercise # 5 from p. 223, Section 5.8. *(Hint: Think about what the conclusion of this problem says about a permutation matrix in terms of the groups we have discussed.)*

10. Do exercise # 7 from p. 364, Section 8.11.

11. What can be said about the eigenvalues of an orthogonal matrix? Give the most precise result you can. *(Hint: Recall that if $A$ is orthogonal, then $AA^t = I$.)*

12. Let $G$ be the group of symmetries of the cube. Write down $3 \times 3$ matrices that represent these symmetries. (Recall that there are 24 such symmetries.) Check your work by verifying that composition of symmetries corresponds to products of matrices for the cases of symmetries of different orders.

13. Find the eigenvalues and eigenvectors for the rotation $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Explain your result geometrically.

14. Define the group $SL(2, \mathbb{Z}_2)$ and give a complete list of its elements. To what more familiar group(s) is this group isomorphic? Explain.

15. Define the groups $GL(n, \mathbb{Z}_p)$ and $SL(n, \mathbb{Z}_p)$, and give examples of elements from each (besides the identity). Find a formula for each group (in terms of $p$) telling how many elements are in the group.

16. We define the **special orthogonal group** to be the group of orthogonal matrices whose determinants are equal to one:

$$SO(n, \mathbb{R}) = \{A \in O(n, \mathbb{R}) | \det(A) = 1\}$$

Give an explicit classification of $SO(2, \mathbb{R})$ in terms of the entries of its elements.