MATH S-15, SUMMER 2001
GROUPS, GRAPHS, AND ALGEBRAIC STRUCTURES FOR COMPUTING
Homework Assignment # 7
Due: July 23, 2001

Reading

In preparation for Monday's lecture, Sections 8.1–8.3 and 8.6–8.8.

Required Problems

1. Consider the matrix $\begin{bmatrix} 1 & x+1 \\ x+1 & x+1 \end{bmatrix}$ from $SL(2, \mathbb{F}_4)$, which is shown in the diagram on page 8 of the notes.
   (a) Let this matrix act five times in succession, starting with the vector $\begin{bmatrix} x \\ x \end{bmatrix}$, and thereby verify that it performs 72° rotations around the pentagon just above the central pentagon in the diagram. To what permutation does it correspond?
   (b) Do the same for $\begin{bmatrix} 0 & x+1 \\ x & x+1 \end{bmatrix}$, starting with $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and thereby determine what is the corresponding rotation of the dodecahedron.

2. Find a 6-element subgroup of $SL(2, \mathbb{F}_4)$ that includes $\begin{bmatrix} x & 1 \\ 0 & x+1 \end{bmatrix}$ and is isomorphic to S_3, the group of symmetries of the equilateral triangle. *(Hint: The given matrix is a rotation about one of the vertices of the top pentagon, involving the three vectors $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ x+1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ x \end{bmatrix}$. You need the three elements of order 2 to complete the subgroup.)*

3. A much smaller group (in fact, a subgroup) than $SL(2, \mathbb{F}_4)$ is $SL(2, \mathbb{Z}_2)$. Write down the six matrices of this group, and show that it is isomorphic to S_3. *(Hint: if you did exploratory exercise # 14 on Homework set 6, then this will be very easy!)*

4. Find a subgroup of 6 matrices in $PSL(2, \mathbb{Z}_5)$ that is isomorphic to S_3. Explicitly identify each matrix in the subgroup with a permutation such as (12) or (123).

5. Find a subgroup of 12 matrices in $SL(2, \mathbb{F}_4)$ that is isomorphic to A_4. Explicitly identify each matrix in the subgroup with a permutation such as (12)(34) or (123). *(Hint: each matrix has the same eigenvector.)*

6. With the two-component vectors over $\mathbb{Z}_5$ arranged into lines as in the notes, i.e.

   Line 1: multiples of $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$

   Line 2: multiples of $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$

   Line 3: multiples of $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$

   Line 4: multiples of $\left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]$

   Line 5: multiples of $\left[\begin{smallmatrix}-2\\1\end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}3\\1\end{smallmatrix}\right]$ if you prefer

   Line 6: multiples of $\left[\begin{smallmatrix}-1\\1\end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}4\\1\end{smallmatrix}\right]$ if you prefer.

   determine the action of each of the following matrices from the group $PSL(2, \mathbb{Z}_5)$ on the lines, and hence associate a permutation with the matrix. You will learn more if you start by determining the eigenvalues and eigenvectors, but you can also just use a brute-force approach and let the matrix operate on a vector from each of the six lines.

   a. $\left[\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right]$

   b. $\left[\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right]$

   c. $\left[\begin{smallmatrix}0&2\\2&0\end{smallmatrix}\right]$

7. With the two-component vectors over $\mathbb{F}_4$ arranged into lines as in the notes, i.e.

   Line 1: multiples of $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$.

   Line 2: multiples of $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$.

   Line 3: multiples of $\left[\begin{smallmatrix}x+1\\1\end{smallmatrix}\right]$.

   Line 4: multiples of $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$.

   Line 5: multiples of $\left[\begin{smallmatrix}x\\1\end{smallmatrix}\right]$.

   determine the action of each of the following matrices from the group $SL(2, \mathbb{F}_4)$ on the lines, and hence associate a permutation with the matrix. You will learn more if you start by determining the eigenvalues and eigenvectors, but you can also just use a brute-force approach and let the matrix operate on a vector from each of the six lines.

   a. $\left[\begin{smallmatrix}x&1\\1&0\end{smallmatrix}\right]$

   b. $\left[\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right]$

   c. $\left[\begin{smallmatrix}x+1&1\\0&x\end{smallmatrix}\right]$

Exploratory Problems

8. Two parts:

   a. Exhibit subgroups of A_5 and of A_6 that are both isomorphic to D_5 and so to one another. (If you use the new version of Groups.exe from the course Web site, this will be very easy!)

   b. Exhibit subgroups of $SL(2, \mathbb{F}_4)$ and $PSL(2, \mathbb{Z}_5)$ that are both isomorphic to D_5 and so to one another. *(Hint: the corresponding permutations that you found in part a tell you how each matrix permutes the lines of vectors.)*

9. Find a 10-element subgroup of $SL(2, \mathbb{F}_4)$ that includes $\left[\begin{smallmatrix} 0 & 1 \\ 1 & x \end{smallmatrix}\right]$ and is isomorphic to D_5, the group of symmetries of the regular pentagon. *(Hint: Consider the* $180°$ *roation about an axis in the equatorial plane that carries* $\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]$ *into itself. You know what this does to* $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ *and* $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$*, so you can write down the matrix. That matrix, along with* $\left[\begin{smallmatrix} 0 & 1 \\ 1 & x \end{smallmatrix}\right]$ *generates the group.)*

10. Find a 12-element subgroup of $SL(2, \mathbb{F}_4)$ that includes $\left[\begin{smallmatrix} x & x+1 \\ x+1 & x \end{smallmatrix}\right]$. *(Hint: This matrix has one eigenvector, and there are 11 others with the same eigenvector: the identity, two more of order 2, and eight of order 3.)*

11. A much smaller group than $SL(2, \mathbb{Z}_5)$ is $SL(2, \mathbb{Z}_3)$. For this group, determine how many elements of order 1, 2, and 3 there are. Show that if you pair up matrices that differ only by an overall sign, you get a group that is isomorphic to the group of symmetries of a regular tetrahedron.

12. Using the same approach as in the notes, determine how many elements are in the group $SL(2, \mathbb{Z}_7)$, then show that the group $PSL(2, \mathbb{Z}_7)$ has 168 elements.

13. Conway's atlas of groups claims that $SL(3, \mathbb{Z}_2)$ has 168 elements, some of which are of order 7. Confirm the number of elements, find one element whose order is 7, and compute its powers.

14. Conway's atlas of groups also claims that $PSL(2, \mathbb{F}_9)$ has 360 elements and is isomorphic to A_6. Show that each of these groups has 360 elements. If you feeling especially ambitious, show that each group has the same number of elements of each order. (Anyone who constructs this isomorphism explicitly will get an automatic 20 out of 20 on Monday's quiz.)