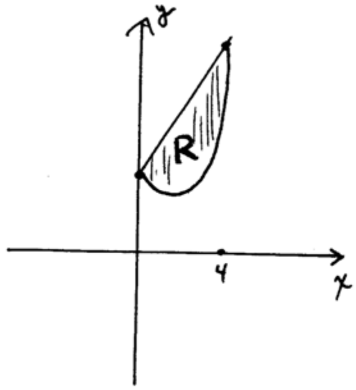
Math S-1ab – Solutions to Third Midterm Exam

(1) Find the <u>area</u> of the region R bounded between the parabola $y = x^2 - x + 5$ and the line $y = 3x + 5$.

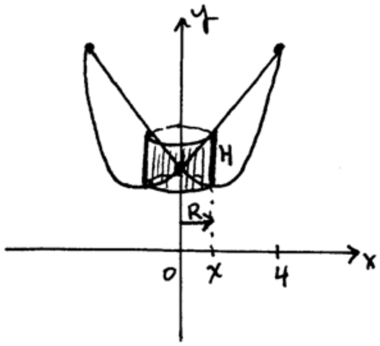


Solution: The vertical distance between an upper graph $y = f(x)$ and a lower graph $y = g(x)$ is given by their difference $f(x) - g(x)$. Thus, the area of a thin vertical strip of width Δx will be $[f(x) - g(x)]\,\Delta x$. Summing these areas and taking a limit, we get the integral $\int_a^b [f(x) - g(x)]\,dx$ where $[a, b]$ is the interval over which the area is being calculated.

For this problem, $f(x) = 3x + 5$ and $g(x) = x^2 - x + 5$. They intersect at $x = 0$ and at $x = 4$, so we get the integral

$$\int_0^4 [(3x+5) - (x^2 - x + 5)]\,dx = \int_0^4 (4x - x^2)\,dx = \cdots = \frac{32}{3}.$$

(2) Find the <u>volume</u> of the solid of revolution obtained by revolving the region R from the previous problem about the y-axis. (The region R lies between the parabola $y = x^2 - x + 5$ and the line $y = 3x + 5$)



Solution: Cylindrical shells are the best choice here for calculating the volume of this solid of revolution. As shown in the picture, the radius of a shell centered around the y-axis will be $R = x$, and the height will be $H = f(x) - g(x)$.

Therefore, we calculate the area of the cylindrical shell as $A = 2\pi RH = 2\pi x[f(x) - g(x)] = 2\pi(4x^2 - x^3) = A(x)$. The volume of the solid can then be calculated by $\int_0^4 A(x)\,dx = \int_0^4 2\pi(4x^2 - x^3)\,dx = \cdots = \frac{128\pi}{3}$.

(3) Find the indicated integrals:

(a) $\int x^2 e^{5x}\,dx$

Solution: Here you want to use integration by parts twice. This will give

$$\int x^2 e^{5x}\,dx = \tfrac{1}{5}x^2 e^{5x} - \tfrac{2}{5}\int x e^{5x}\,dx = \tfrac{1}{5}x^2 e^{5x} - \tfrac{2}{5}[\tfrac{1}{5}x e^{5x} - \tfrac{1}{25}\int e^{5x}\,dx]$$

$$= \tfrac{1}{5}x^2 e^{5x} - \tfrac{2}{5}[\tfrac{1}{5}x e^{5x} - \tfrac{1}{125}e^{5x}] = (\tfrac{1}{5}x^2 - \tfrac{2}{25}x + \tfrac{2}{125})e^{5x} + C$$

(b) $\int \frac{4}{(1-25x^2)^{3/2}}\,dx$

Solution: Here you want to use the trigonometric substitution: $5x = \sin\theta$. This gives $x = \tfrac{1}{5}\sin\theta$, $dx = \tfrac{1}{5}\cos\theta\,d\theta$, $\sqrt{1-25x^2} = \cos\theta$.

Therefore, the integral becomes $\int \frac{4}{(1-25x^2)^{3/2}}\,dx = \int \frac{4\,\tfrac{1}{5}\cos\theta\,d\theta}{\cos^3\theta}$

$$= \frac{4}{5}\int \sec^2\theta\,d\theta = \frac{4}{5}\tan\theta + C = \frac{5x}{\sqrt{1-25x^2}} + C$$

(c) $\int_3^{+\infty} \frac{dx}{(5x+1)^{3/2}}$

Solution: This is an improper integral, so we compute as follows:

$$\int_3^{+\infty} \frac{dx}{(5x+1)^{3/2}} = \lim_{T\to\infty}\int_3^T \frac{dx}{(5x+1)^{3/2}} = \lim_{T\to\infty}\left[\frac{-2}{5\sqrt{5x+1}}\right]_3^T$$

$$= \lim_{T\to\infty}\left[\frac{1}{10} - \frac{-2}{5\sqrt{5T+1}}\right] = \frac{1}{10}$$

(3d) $\int_4^7 \frac{(8x-3)\,dx}{(2x+1)(x-3)}$

Solution: Here you'll want to find a partial fractions decomposition for the rational function in the integrand. This gives

$$\int_4^7 \frac{(8x-3)\,dx}{(2x+1)(x-3)} = \int_4^7 \left[\frac{2}{2x+1} + \frac{3}{x-3}\right] dx$$

$$= [\ln|2x+1| + \ln|x-3|]_{x=4}^{x=7} = \cdots = \ln 5 - \ln 3 + 6\ln 2$$

(4) Solve the differential equation $\frac{dy}{dx} - y\sin x = \sin x$, subject to the initial condition $y(0) = 4$.

Solution: You can view this as a <u>separable differential equation</u> or as a <u>first order linear differential equation</u>. As a separable equation, we can rewrite it as $\frac{dy}{dx} = (1+y)\sin x$. This gives $\int \frac{dy}{1+y} = \int \sin x dx$. This integrates to yield (watch those arbitrary constants!) $\ln|1+y| = -\cos x + C$ or, after exponentiating and renaming the constants, $1+y = Ae^{-\cos x}$ where A could be positive or negative. If we now use the initial condition that $y = 4$ when $x = 0$, we get $5 = \frac{A}{e}$ or $A = 5e$. Therefore, we have

$y(x) = 5e\,e^{-\cos x} - 1 = 5\,e^{1-\cos x} - 1$.

If you chose to proceed by finding an integrating factor, that factor should be $e^{\cos x}$.

(5) Consider a leaky 20-liter container kept full of sugar water by adding pure water at a rate of 4 liters per minute and very concentrated sugar water at a rate of 2 liters per minute. The concentrated sugar water has a concentration of 50 grams per liter. The leaky container initially has a concentration of 10 grams per liter and is kept constantly stirred as liquid is added.

(a) Write down a differential equation for the amount of sugar $x(t)$ in the container after t minutes. Specify what the initial condition for your equation is.

(b) Solve for $x(t)$ for the given initial condition.

(c) What will be the concentration of the sugar in the container after $t = \ln(32)$ minutes? (That about 3 minutes, 28 seconds.)

Solution: First of all, the wording of the problem was not as intended and the algebra became more awkward as a result. Consequently, let's just ignore part (c). For part (a), the key things to notice are that salt is going in at a rate of (50 gm/liter)(2 liters/min) = 100 gm/min. At any given moment, there are x grams of salt in the container at a concentration of $\frac{x \text{ grams}}{20 \text{ liters}} = \frac{x}{20}$ gm/liter. The total volume of liquid exiting the container is 6 liters/min, so this gives ($\frac{x}{20}$ gm/liter)(6 liters/min) = 0.3 gm/min of salt leaving the container. Putting these together, we get $\frac{dx}{dt} = 100 - .3x$. This is separable and we can solve to get $1000 - 3x = Ae^{-0.3t}$. The initial amount of salt in the container is the product of the initial concentration and the volume of the container. That is, $x(0) = 200$ grams. Substituting this into the solution, we get that $A = 400$. Putting it all together we get that $x(t) = \frac{1000 - 400e^{-0.3t}}{3}$.