

Harvard Summer School -- Math S-1ab Practice Final Exam

- (1) Decide whether each of the following series converge absolutely, converge conditionally, or diverge. Explain why you made your decision and show your work.

(a)
$$\sum_{k=0}^{\infty} (-1)^k \frac{2^{2k+1}}{(k!)^2}$$

(b)
$$\sum_{k=2}^{\infty} (-1)^k \frac{k^2}{2(k^2 - 1)}$$

(c)
$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^2 + 4}}$$

- (2) (a) Write the repeating decimal expression .023023023... as a rational number (ratio of integers) by expressing it as a geometric series and computing the sum of that series.

- (b) Show whether or not the integral $\int_2^{\infty} \frac{x^2 - 3}{2x^4 + 7x - 2} dx$ converges to a finite number by comparison with a series.

- (3)(a) Find the interval of convergence for the following power series: $\sum_{k=1}^{\infty} (-1)^{2k} \frac{(2x - 5)^{2k}}{k}$

- (b) Use the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and a known Maclaurin series to derive the Maclaurin series expansion for $\sin^2 x$. Express your answer in sigma (Σ) notation.

- (4a) What approximate value do you get for $\ln(1.05)$ by using up to the third order terms of the Maclaurin series for $\ln(1 + x)$?

- (b) Give an upper bound for the error in this approximation, using the estimate from Taylor's Theorem.

- (c) What is the least number of terms of this series that would be necessary in order to guarantee accuracy within an error of 0.001?

- (5) Find the indicated integrals:

(a)
$$\int \frac{4}{\sqrt{9x^2 + 25}} dx$$

(b)
$$\int_3^{+\infty} \frac{dx}{(5x+1)^{3/2}}$$

(c)
$$\int x^2 \cos 5x dx$$

(6) Find the volume of the solid of revolution obtained by revolving about the y -axis the region bounded between the line $y = 5$ and the parabola $y = 9 - x^2$.

(7) Consider the function:

$$f(x) = (1 + 2x)^{\frac{3}{x}}$$

a) Find $\lim_{x \rightarrow 0^+} f(x)$

b) Find $f'(1)$

(8) Sketch the graph of $y = \frac{x^2 - x}{(x + 1)^2}$. Plot any intercepts, stationary points, and points of inflection. Show

any horizontal and vertical asymptotes. [The derivatives are $\frac{dy}{dx} = \frac{3x - 1}{(x + 1)^3}$ and $\frac{d^2y}{dx^2} = \frac{-6x + 6}{(x + 1)^4}$.]

(9) A farmer wishes to fence a rectangular pasture having a total area of 12,000 square meters. She wants to divide it into two parts with a fence across the middle. Fencing around the outside costs \$7.50 per meter, but the farmer can use less expensive fencing at \$3 per meter as the divider. What dimensions will result in the least cost?

(10) A rapid transit trolley moves with a constant acceleration and covers the distance between two points 300 feet apart in 8 seconds. Its velocity as it passes the second point is 50 feet per second.

(a) What is its acceleration? $a =$ _____

(b) What is the velocity of the trolley as it passes the first point? $v_0 =$ _____